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# Mathematical Modeling of Control Systems

## 2–1 INTRODUCTION

In studying control systems the reader must be able to model dynamic systems in mathematical terms and analyze their dynamic characteristics. A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on one's perspective.

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system—for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving reasonable mathematical models is the most important part of the entire analysis of control systems.

Throughout this book we assume that the principle of causality applies to the systems considered. This means that the current output of the system (the output at time $t = 0$) depends on the past input (the input for $t < 0$) but does not depend on the future input (the input for $t > 0$).

**Mathematical Models.** Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations. On the other hand, for the transient-response or frequency-response analysis of single-input, single-output, linear, time-invariant systems, the transfer-function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

**Simplicity Versus Accuracy.** In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis. In deriving a reasonably simplified mathematical model, we frequently find it necessary to ignore certain inherent physical properties of the system. In particular, if a linear lumped-parameter mathematical model (that is, one employing ordinary differential equations) is desired, it is always necessary to ignore certain nonlinearities and distributed parameters that may be present in the physical system. If the effects that these ignored properties have on the response are small, good agreement will be obtained between the results of the analysis of a mathematical model and the results of the experimental study of the physical system.

In general, in solving a new problem, it is desirable to build a simplified model so that we can get a general feeling for the solution. A more complete mathematical model may then be built and used for a more accurate analysis.

We must be well aware that a linear lumped-parameter model, which may be valid in low-frequency operations, may not be valid at sufficiently high frequencies, since the neglected property of distributed parameters may become an important factor in the dynamic behavior of the system. For example, the mass of a spring may be neglected in low-frequency operations, but it becomes an important property of the system at high frequencies. (For the case where a mathematical model involves considerable errors, robust control theory may be applied. Robust control theory is presented in Chapter 10.)

**Linear Systems.** A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. It is this principle that allows one to build up complicated solutions to the linear differential equation from simple solutions.

In an experimental investigation of a dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered linear.

**Linear Time-Invariant Systems and Linear Time-Varying Systems.** A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant differential equations—that is, constant-coefficient differential equations. Such systems are called *linear time-invariant* (or *linear constant-coefficient*) systems. Systems that are represented by differential equations whose coefficients are functions of time are called *linear time-varying* systems. An example of a time-varying control system is a spacecraft control system. (The mass of a spacecraft changes due to fuel consumption.)

**Outline of the Chapter.** Section 2–1 has presented an introduction to the mathematical modeling of dynamic systems. Section 2–2 presents the transfer function and impulse-response function. Section 2–3 introduces automatic control systems and Section 2–4 discusses concepts of modeling in state space. Section 2–5 presents state-space representation of dynamic systems. Section 2–6 discusses transformation of mathematical models with MATLAB. Finally, Section 2–7 discusses linearization of nonlinear mathematical models.

## 2–2 TRANSFER FUNCTION AND IMPULSE-RESPONSE FUNCTION

In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equations. We begin by defining the transfer function and follow with a derivation of the transfer function of a differential equation system. Then we discuss the impulse-response function.

**Transfer Function.** The *transfer function* of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where $y$ is the output of the system and $x$ is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left. \frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]} \right|_{\text{zero initial conditions}}$$
$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in $s$. If the highest power of $s$ in the denominator of the transfer function is equal to $n$, the system is called an *nth-order system*.

**Comments on Transfer Function.** The applicability of the concept of the transfer function is limited to linear, time-invariant, differential equation systems. The transfer function approach, however, is extensively used in the analysis and design of such systems. In what follows, we shall list important comments concerning the transfer function. (Note that a system referred to in the list is one described by a linear, time-invariant, differential equation.)

1. The transfer function of a system is a mathematical model in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
2. The transfer function is a property of a system itself, independent of the magnitude and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)
4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.
5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

**Convolution Integral.** For a linear, time-invariant system the transfer function $G(s)$ is

$$G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ is the Laplace transform of the input to the system and $Y(s)$ is the Laplace transform of the output of the system, where we assume that all initial conditions involved are zero. It follows that the output $Y(s)$ can be written as the product of $G(s)$ and $X(s)$, or

$$Y(s) = G(s)X(s) \tag{2–1}$$

Note that multiplication in the complex domain is equivalent to convolution in the time domain (see Appendix A), so the inverse Laplace transform of Equation (2–1) is given by the following convolution integral:

$$\begin{aligned} y(t) &= \int_0^t x(\tau)g(t - \tau)\,d\tau \\ &= \int_0^t g(\tau)x(t - \tau)\,d\tau \end{aligned}$$

where both $g(t)$ and $x(t)$ are 0 for $t < 0$.

**Impulse-Response Function.** Consider the output (response) of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is

$$Y(s) = G(s) \tag{2–2}$$

The inverse Laplace transform of the output given by Equation (2–2) gives the impulse response of the system. The inverse Laplace transform of $G(s)$, or

$$\mathscr{L}^{-1}[G(s)] = g(t)$$

is called the impulse-response function. This function $g(t)$ is also called the weighting function of the system.

The impulse-response function $g(t)$ is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function. Therefore, the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response. (In practice, a pulse input with a very short duration compared with the significant time constants of the system can be considered an impulse.)

## 2–3 AUTOMATIC CONTROL SYSTEMS

A control system may consist of a number of components. To show the functions performed by each component, in control engineering, we commonly use a diagram called the *block diagram*. This section first explains what a block diagram is. Next, it discusses introductory aspects of automatic control systems, including various control actions. Then, it presents a method for obtaining block diagrams for physical systems, and, finally, discusses techniques to simplify such diagrams.

**Block Diagrams.** A *block diagram* of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. Differing from a purely abstract mathematical representation, a block diagram has the advantage of indicating more realistically the signal flows of the actual system.

In a block diagram all system variables are linked to each other through functional blocks. The *functional* block or simply *block* is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that the signal can pass only in the direction of the arrows. Thus a block diagram of a control system explicitly shows a unilateral property.

Figure 2–1 shows an element of the block diagram. The arrowhead pointing toward the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as *signals*.

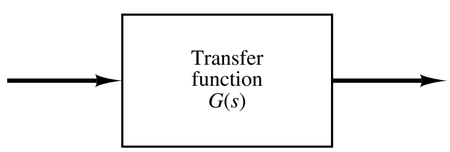


**Figure 2–1** Element of a block diagram.

Note that the dimension of the output signal from the block is the dimension of the input signal multiplied by the dimension of the transfer function in the block.

The advantages of the block diagram representation of a system are that it is easy to form the overall block diagram for the entire system by merely connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

In general, the functional operation of the system can be visualized more readily by examining the block diagram than by examining the physical system itself. A block diagram contains information concerning dynamic behavior, but it does not include any information on the physical construction of the system. Consequently, many dissimilar and unrelated systems can be represented by the same block diagram.

It should be noted that in a block diagram the main source of energy is not explicitly shown and that the block diagram of a given system is not unique. A number of different block diagrams can be drawn for a system, depending on the point of view of the analysis.

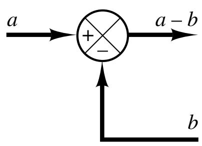


**Figure 2–2**
Summing point.

***Summing Point.*** Referring to Figure 2–2, a circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

***Branch Point.*** A *branch point* is a point from which the signal from a block goes concurrently to other blocks or summing points.

**Block Diagram of a Closed-Loop System.** Figure 2–3 shows an example of a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point, where it is compared with the reference input $R(s)$. The closed-loop nature of the system is clearly indicated by the figure. The output of the block, $C(s)$ in this case, is obtained by multiplying the transfer function $G(s)$ by the input to the block, $E(s)$. Any linear control system may be represented by a block diagram consisting of blocks, summing points, and branch points.

When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. For example, in a temperature control system, the output signal is usually the controlled temperature. The output signal, which has the dimension of temperature, must be converted to a force or position or voltage before it can be compared with the input signal. This conversion is accomplished by the feedback element whose transfer function is $H(s)$, as shown in Figure 2–4. The role of the feedback element is to modify the output before it is compared with the input. (In most cases the feedback element is a sensor that measures

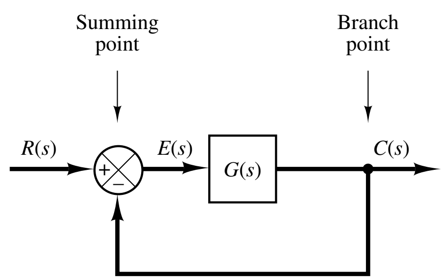


**Figure 2–3**
Block diagram of a closed-loop system.

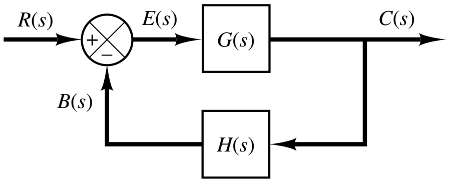


**Figure 2–4**
Closed-loop system.

the output of the plant. The output of the sensor is compared with the system input, and the actuating error signal is generated.) In the present example, the feedback signal that is fed back to the summing point for comparison with the input is $B(s) = H(s)C(s)$.

**Open-Loop Transfer Function and Feedforward Transfer Function.** Referring to Figure 2–4, the ratio of the feedback signal $B(s)$ to the actuating error signal $E(s)$ is called the *open-loop transfer function*. That is,

$$\text{Open-loop transfer function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

The ratio of the output $C(s)$ to the actuating error signal $E(s)$ is called the *feedforward transfer function*, so that

$$\text{Feedforward transfer function} = \frac{C(s)}{E(s)} = G(s)$$

If the feedback transfer function $H(s)$ is unity, then the open-loop transfer function and the feedforward transfer function are the same.

**Closed-Loop Transfer Function.** For the system shown in Figure 2–4, the output $C(s)$ and input $R(s)$ are related as follows: since

$$\begin{aligned} C(s) &= G(s)E(s) \\ E(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s) \end{aligned}$$

eliminating $E(s)$ from these equations gives

$$C(s) = G(s)\left[R(s) - H(s)C(s)\right]$$

or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{2–3}$$

The transfer function relating $C(s)$ to $R(s)$ is called the *closed-loop transfer function*. It relates the closed-loop system dynamics to the dynamics of the feedforward elements and feedback elements.

From Equation (2–3), $C(s)$ is given by

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Thus the output of the closed-loop system clearly depends on both the closed-loop transfer function and the nature of the input.

**Obtaining Cascaded, Parallel, and Feedback (Closed-Loop) Transfer Functions with MATLAB.** In control-systems analysis, we frequently need to calculate the cascaded transfer functions, parallel-connected transfer functions, and feedback-connected (closed-loop) transfer functions. MATLAB has convenient commands to obtain the cascaded, parallel, and feedback (closed-loop) transfer functions.

Suppose that there are two components $G_1(s)$ and $G_2(s)$ connected differently as shown in Figure 2–5 (a), (b), and (c), where

$$G_1(s) = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{\text{num2}}{\text{den2}}$$

To obtain the transfer functions of the cascaded system, parallel system, or feedback (closed-loop) system, the following commands may be used:

```
[num, den] = series(num1,den1,num2,den2)
[num, den] = parallel(num1,den1,num2,den2)
[num, den] = feedback(num1,den1,num2,den2)
```

As an example, consider the case where

$$G_1(s) = \frac{10}{s^2 + 2s + 10} = \frac{\text{num1}}{\text{den1}}, \qquad G_2(s) = \frac{5}{s + 5} = \frac{\text{num2}}{\text{den2}}$$

MATLAB Program 2–1 gives $C(s)/R(s) = \text{num/den}$ for each arrangement of $G_1(s)$ and $G_2(s)$. Note that the command

```
printsys(num,den)
```

displays the num/den $\left[\text{that is, the transfer function } C(s)/R(s)\right]$ of the system considered.

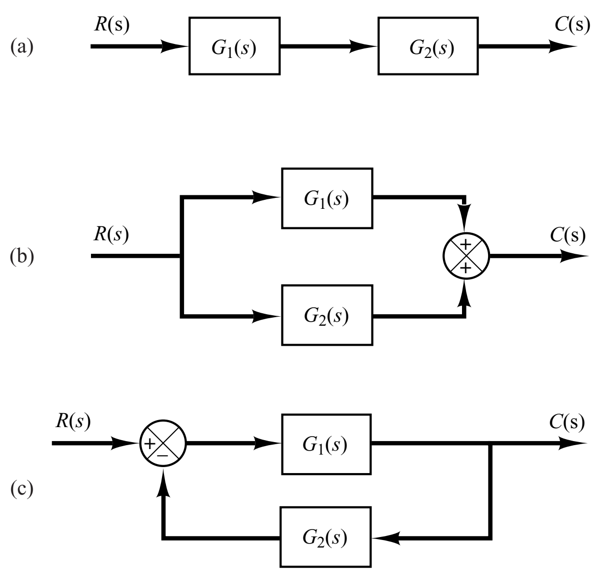


**Figure 2–5**
(a) Cascaded system; (b) parallel system; (c) feedback (closed-loop) system.

**MATLAB Program 2–1**

```
num1 = [10];
den1 = [1  2  10];
num2 = [5];
den2 = [1  5];
[num, den] = series(num1,den1,num2,den2);
printsys(num,den)

num/den =

            50
    -----------------------
    s^3 + 7s^2 + 20s + 50

[num, den] = parallel(num1,den1,num2,den2);
printsys(num,den)

num/den =

     5s^2 + 20s + 100
    -----------------------
    s^3 + 7s^2 + 20s + 50

[num, den] = feedback(num1,den1,num2,den2);
printsys(num,den)

num/den =

          10s + 50
    ------------------------
    s^3 + 7s^2 + 20s + 100
```

**Automatic Controllers.** An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value. The manner in which the automatic controller produces the control signal is called the *control action*. Figure 2–6 is a block diagram of an industrial control system, which

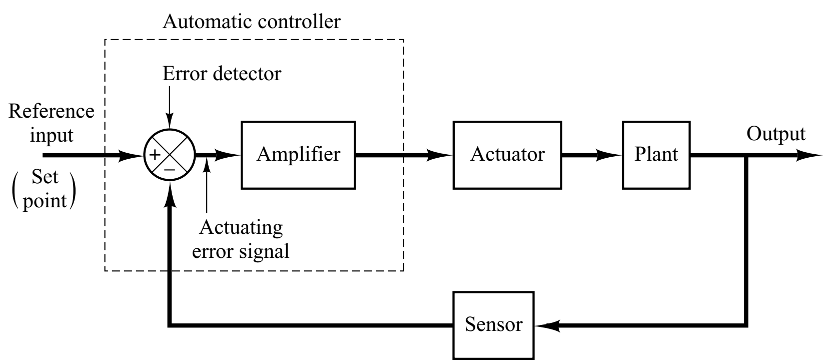


**Figure 2–6** Block diagram of an industrial control system, which consists of an automatic controller, an actuator, a plant, and a sensor (measuring element).

consists of an automatic controller, an actuator, a plant, and a sensor (measuring element). The controller detects the actuating error signal, which is usually at a very low power level, and amplifies it to a sufficiently high level. The output of an automatic controller is fed to an actuator, such as an electric motor, a hydraulic motor, or a pneumatic motor or valve. (The actuator is a power device that produces the input to the plant according to the control signal so that the output signal will approach the reference input signal.)

The sensor or measuring element is a device that converts the output variable into another suitable variable, such as a displacement, pressure, voltage, etc., that can be used to compare the output to the reference input signal. This element is in the feedback path of the closed-loop system. The set point of the controller must be converted to a reference input with the same units as the feedback signal from the sensor or measuring element.

**Classifications of Industrial Controllers.** Most industrial controllers may be classified according to their control actions as:

1. Two-position or on–off controllers
2. Proportional controllers
3. Integral controllers
4. Proportional-plus-integral controllers
5. Proportional-plus-derivative controllers
6. Proportional-plus-integral-plus-derivative controllers

Most industrial controllers use electricity or pressurized fluid such as oil or air as power sources. Consequently, controllers may also be classified according to the kind of power employed in the operation, such as pneumatic controllers, hydraulic controllers, or electronic controllers. What kind of controller to use must be decided based on the nature of the plant and the operating conditions, including such considerations as safety, cost, availability, reliability, accuracy, weight, and size.

**Two-Position or On–Off Control Action.** In a two-position control system, the actuating element has only two fixed positions, which are, in many cases, simply on and off. Two-position or on–off control is relatively simple and inexpensive and, for this reason, is very widely used in both industrial and domestic control systems.

Let the output signal from the controller be $u(t)$ and the actuating error signal be $e(t)$. In two-position control, the signal $u(t)$ remains at either a maximum or minimum value, depending on whether the actuating error signal is positive or negative, so that

$$\begin{aligned} u(t) &= U_1, \qquad \text{for } e(t) > 0 \\ &= U_2, \qquad \text{for } e(t) < 0 \end{aligned}$$

where $U_1$ and $U_2$ are constants. The minimum value $U_2$ is usually either zero or $-U_1$. Two-position controllers are generally electrical devices, and an electric solenoid-operated valve is widely used in such controllers. Pneumatic proportional controllers with very high gains act as two-position controllers and are sometimes called pneumatic two-position controllers.

Figures 2–7(a) and (b) show the block diagrams for two-position or on–off controllers. The range through which the actuating error signal must move before the switching occurs

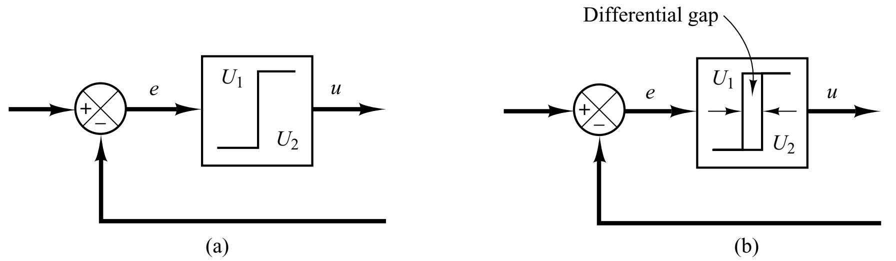


**Figure 2–7**
(a) Block diagram of an on–off controller; (b) block diagram of an on–off controller with differential gap.

is called the *differential gap*. A differential gap is indicated in Figure 2–7(b). Such a differential gap causes the controller output $u(t)$ to maintain its present value until the actuating error signal has moved slightly beyond the zero value. In some cases, the differential gap is a result of unintentional friction and lost motion; however, quite often it is intentionally provided in order to prevent too-frequent operation of the on–off mechanism.

Consider the liquid-level control system shown in Figure 2–8(a), where the electromagnetic valve shown in Figure 2–8(b) is used for controlling the inflow rate. This valve is either open or closed. With this two-position control, the water inflow rate is either a positive constant or zero. As shown in Figure 2–9, the output signal continuously moves between the two limits required to cause the actuating element to move from one fixed position to the other. Notice that the output curve follows one of two exponential curves, one corresponding to the filling curve and the other to the emptying curve. Such output oscillation between two limits is a typical response characteristic of a system under two-position control.

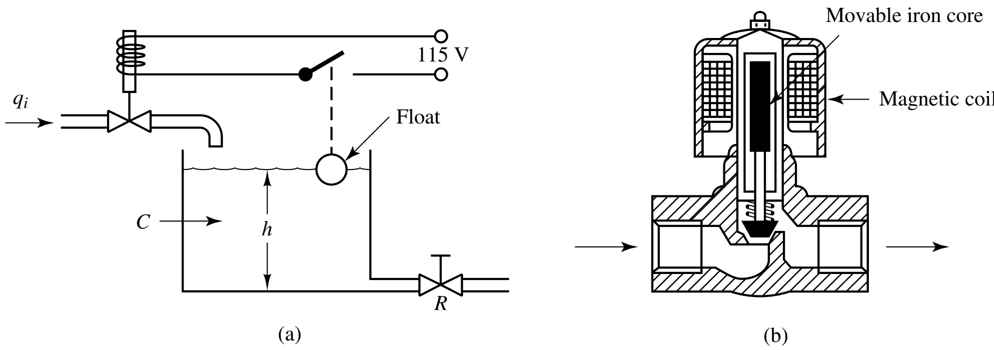


**Figure 2–8**
(a) Liquid-level control system; (b) electromagnetic valve.

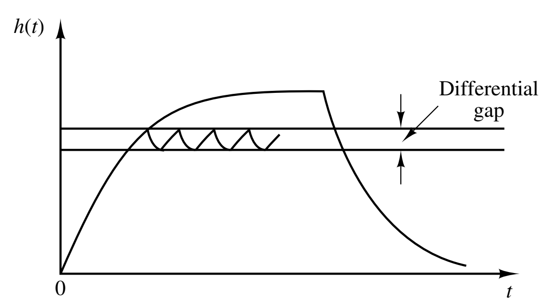


**Figure 2–9**
Level $h(t)$-versus-$t$ curve for the system shown in Figure 2–8(a).

From Figure 2–9, we notice that the amplitude of the output oscillation can be reduced by decreasing the differential gap. The decrease in the differential gap, however, increases the number of on–off switchings per minute and reduces the useful life of the component. The magnitude of the differential gap must be determined from such considerations as the accuracy required and the life of the component.

**Proportional Control Action.** For a controller with proportional control action, the relationship between the output of the controller $u(t)$ and the actuating error signal $e(t)$ is

$$u(t) = K_p e(t)$$

or, in Laplace-transformed quantities,

$$\frac{U(s)}{E(s)} = K_p$$

where $K_p$ is termed the proportional gain.

Whatever the actual mechanism may be and whatever the form of the operating power, the proportional controller is essentially an amplifier with an adjustable gain.

**Integral Control Action.** In a controller with integral control action, the value of the controller output $u(t)$ is changed at a rate proportional to the actuating error signal $e(t)$. That is,

$$\frac{du(t)}{dt} = K_i e(t)$$

or

$$u(t) = K_i \int_0^t e(t)\,dt$$

where $K_i$ is an adjustable constant. The transfer function of the integral controller is

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

**Proportional-Plus-Integral Control Action.** The control action of a proportional-plus-integral controller is defined by

$$u(t) = K_p e(t) + \frac{K_p}{T_i} \int_0^t e(t)\,dt$$

or the transfer function of the controller is

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

where $T_i$ is called the *integral time*.

**Proportional-Plus-Derivative Control Action.** The control action of a proportional-plus-derivative controller is defined by

$$u(t) = K_p e(t) + K_p T_d \frac{de(t)}{dt}$$

and the transfer function is

$$\frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

where $T_d$ is called the *derivative time*.

**Proportional-Plus-Integral-Plus-Derivative Control Action.** The combination of proportional control action, integral control action, and derivative control action is termed proportional-plus-integral-plus-derivative control action. It has the advantages of each of the three individual control actions. The equation of a controller with this combined action is given by

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)\,dt + K_p T_d \frac{de(t)}{dt}$$

or the transfer function is

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

where $K_p$ is the proportional gain, $T_i$ is the integral time, and $T_d$ is the derivative time. The block diagram of a proportional-plus-integral-plus-derivative controller is shown in Figure 2–10.

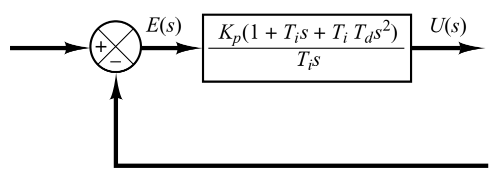


**Figure 2–10**
Block diagram of a proportional-plus-integral-plus-derivative controller.

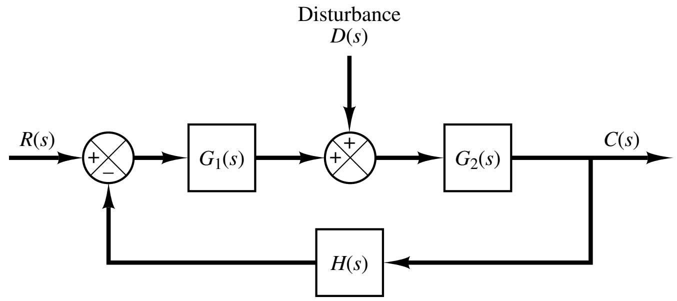


**Figure 2–11** Closed-loop system subjected to a disturbance.

**Closed-Loop System Subjected to a Disturbance.** Figure 2–11 shows a closed-loop system subjected to a disturbance. When two inputs (the reference input and disturbance) are present in a linear time-invariant system, each input can be treated independently of the other; and the outputs corresponding to each input alone can be added to give the complete output. The way each input is introduced into the system is shown at the summing point by either a plus or minus sign.

Consider the system shown in Figure 2–11. In examining the effect of the disturbance $D(s)$, we may assume that the reference input is zero; we may then calculate the response $C_D(s)$ to the disturbance only. This response can be found from

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

On the other hand, in considering the response to the reference input $R(s)$, we may assume that the disturbance is zero. Then the response $C_R(s)$ to the reference input $R(s)$ can be obtained from

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

The response to the simultaneous application of the reference input and disturbance can be obtained by adding the two individual responses. In other words, the response $C(s)$ due to the simultaneous application of the reference input $R(s)$ and disturbance $D(s)$ is given by

$$\begin{aligned} C(s) &= C_R(s) + C_D(s) \\ &= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + D(s)\right] \end{aligned}$$

Consider now the case where $|G_1(s)H(s)| \gg 1$ and $|G_1(s)G_2(s)H(s)| \gg 1$. In this case, the closed-loop transfer function $C_D(s)/D(s)$ becomes almost zero, and the effect of the disturbance is suppressed. This is an advantage of the closed-loop system.

On the other hand, the closed-loop transfer function $C_R(s)/R(s)$ approaches $1/H(s)$ as the gain of $G_1(s)G_2(s)H(s)$ increases. This means that if $|G_1(s)G_2(s)H(s)| \gg 1$, then the closed-loop transfer function $C_R(s)/R(s)$ becomes independent of $G_1(s)$ and $G_2(s)$ and inversely proportional to $H(s)$, so that the variations of $G_1(s)$ and $G_2(s)$ do not affect the closed-loop transfer function $C_R(s)/R(s)$. This is another advantage of the closed-loop system. It can easily be seen that any closed-loop system with unity feedback, $H(s) = 1$, tends to equalize the input and output.

**Procedures for Drawing a Block Diagram.** To draw a block diagram for a system, first write the equations that describe the dynamic behavior of each component. Then take the Laplace transforms of these equations, assuming zero initial conditions, and represent each Laplace-transformed equation individually in block form. Finally, assemble the elements into a complete block diagram.

As an example, consider the *RC* circuit shown in Figure 2–12(a). The equations for this circuit are

$$i = \frac{e_i - e_o}{R} \tag{2–4}$$

$$e_o = \frac{\int i\,dt}{C} \tag{2–5}$$

The Laplace transforms of Equations (2–4) and (2–5), with zero initial condition, become

$$I(s) = \frac{E_i(s) - E_o(s)}{R} \tag{2–6}$$

$$E_o(s) = \frac{I(s)}{Cs} \tag{2–7}$$

Equation (2–6) represents a summing operation, and the corresponding diagram is shown in Figure 2–12(b). Equation (2–7) represents the block as shown in Figure 2–12(c). Assembling these two elements, we obtain the overall block diagram for the system as shown in Figure 2–12(d).

**Block Diagram Reduction.** It is important to note that blocks can be connected in series only if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block.

Any number of cascaded blocks representing nonloading components can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions.

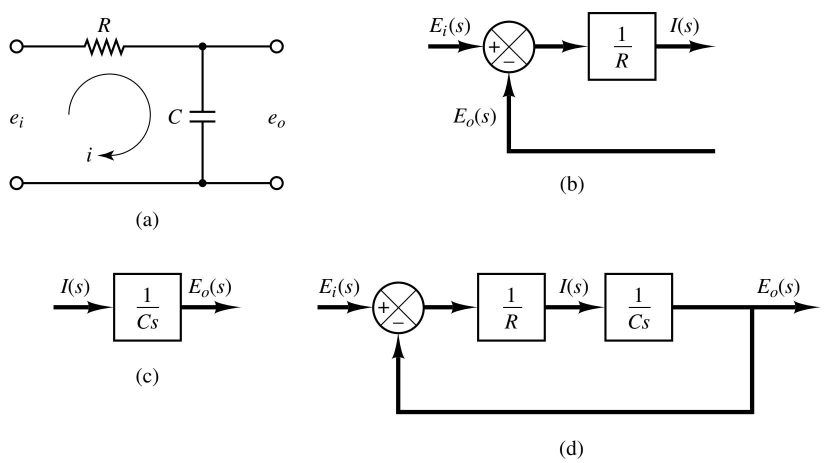


**Figure 2–12**
(a) *RC* circuit;
(b) block diagram representing Equation (2–6);
(c) block diagram representing Equation (2–7);
(d) block diagram of the *RC* circuit.

A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangement. Simplification of the block diagram by rearrangements considerably reduces the labor needed for subsequent mathematical analysis. It should be noted, however, that as the block diagram is simplified, the transfer functions in new blocks become more complex because new poles and new zeros are generated.

**EXAMPLE 2–1** Consider the system shown in Figure 2–13(a). Simplify this diagram.

By moving the summing point of the negative feedback loop containing $H_2$ outside the positive feedback loop containing $H_1$, we obtain Figure 2–13(b). Eliminating the positive feedback loop, we have Figure 2–13(c). The elimination of the loop containing $H_2/G_1$ gives Figure 2–13(d). Finally, eliminating the feedback loop results in Figure 2–13(e).

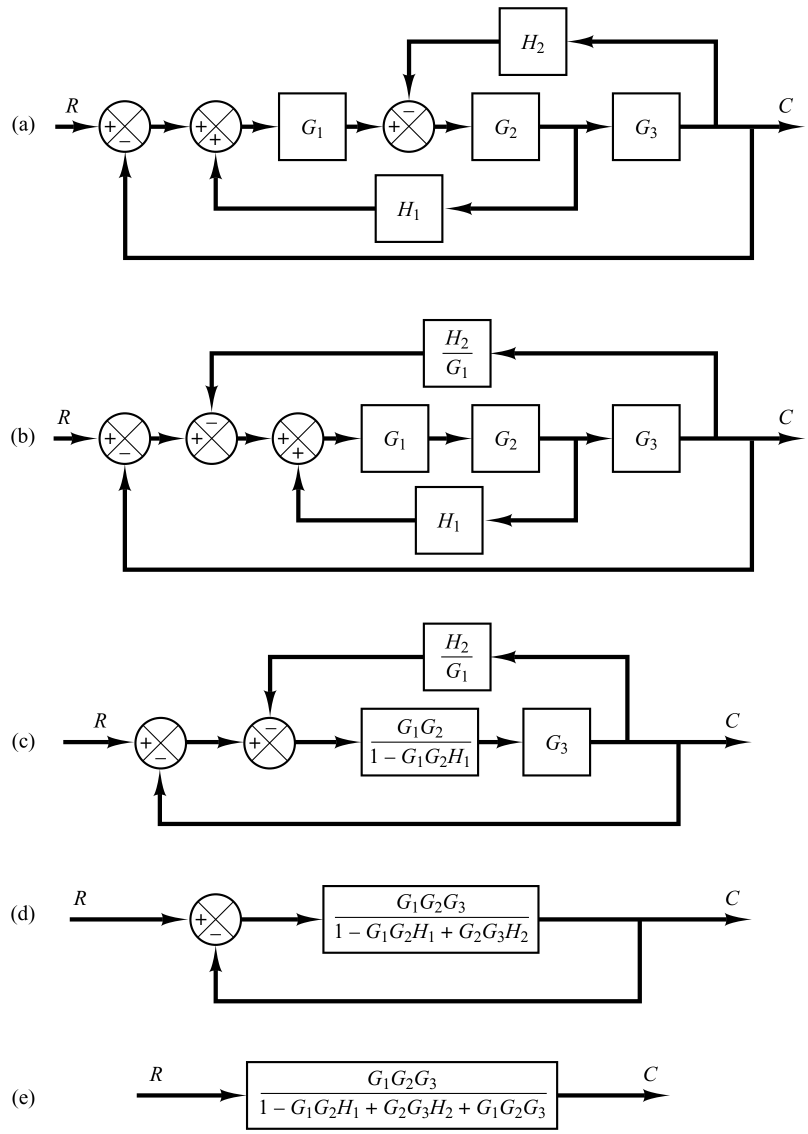


**Figure 2–13**
(a) Multiple-loop system;
(b)–(e) successive reductions of the block diagram shown in (a).

Notice that the numerator of the closed-loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feedforward path. The denominator of $C(s)/R(s)$ is equal to

$$1 + \sum (\text{product of the transfer functions around each loop})$$
$$= 1 + (-G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3)$$
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

(The positive feedback loop yields a negative term in the denominator.)

## 2–4 MODELING IN STATE SPACE

In this section we shall present introductory material on state-space analysis of control systems.

**Modern Control Theory.** The modern trend in engineering systems is toward greater complexity, due mainly to the requirements of complex tasks and good accuracy. Complex systems may have multiple inputs and multiple outputs and may be time varying. Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity, and easy access to large scale computers, modern control theory, which is a new approach to the analysis and design of complex control systems, has been developed since around 1960. This new approach is based on the concept of state. The concept of state by itself is not new, since it has been in existence for a long time in the field of classical dynamics and other fields.

**Modern Control Theory Versus Conventional Control Theory.** Modern control theory is contrasted with conventional control theory in that the former is applicable to multiple-input, multiple-output systems, which may be linear or nonlinear, time invariant or time varying, while the latter is applicable only to linear time-invariant single-input, single-output systems. Also, modern control theory is essentially time-domain approach and frequency domain approach (in certain cases such as H-infinity control), while conventional control theory is a complex frequency-domain approach. Before we proceed further, we must define state, state variables, state vector, and state space.

**State.** The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Note that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

**State Variables.** The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least $n$ variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Note that state variables need not be physically measurable or observable quantities. Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state-space methods. Practically, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

**State Vector.** If $n$ state variables are needed to completely describe the behavior of a given system, then these $n$ state variables can be considered the $n$ components of a vector $\mathbf{x}$. Such a vector is called a *state vector*. A state vector is thus a vector that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

**State Space.** The $n$-dimensional space whose coordinate axes consist of the $x_1$ axis, $x_2$ axis, $\ldots$, $x_n$ axis, where $x_1, x_2, \ldots, x_n$ are state variables, is called a *state space*. Any state can be represented by a point in the state space.

**State-Space Equations.** In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. As we shall see in Section 2–5, the state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$. Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus the outputs of integrators serve as state variables. The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

Assume that a multiple-input, multiple-output system involves $n$ integrators. Assume also that there are $r$ inputs $u_1(t), u_2(t), \ldots, u_r(t)$ and $m$ outputs $y_1(t), y_2(t), \ldots, y_m(t)$. Define $n$ outputs of the integrators as state variables: $x_1(t), x_2(t), \ldots, x_n(t)$ Then the system may be described by

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\;\;\vdots \\
\dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
\tag{2–8}
$$

The outputs $y_1(t), y_2(t), \ldots, y_m(t)$ of the system may be given by

$$\begin{aligned} y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ &\;\;\vdots \\ y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{aligned} \tag{2–9}$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

then Equations (2–8) and (2–9) become

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{2–10}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{2–11}$$

where Equation (2–10) is the state equation and Equation (2–11) is the output equation. If vector functions **f** and/or **g** involve time $t$ explicitly, then the system is called a time-varying system.

If Equations (2–10) and (2–11) are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{2–12}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{2–13}$$

where $\mathbf{A}(t)$ is called the state matrix, $\mathbf{B}(t)$ the input matrix, $\mathbf{C}(t)$ the output matrix, and $\mathbf{D}(t)$ the direct transmission matrix. (Details of linearization of nonlinear systems about

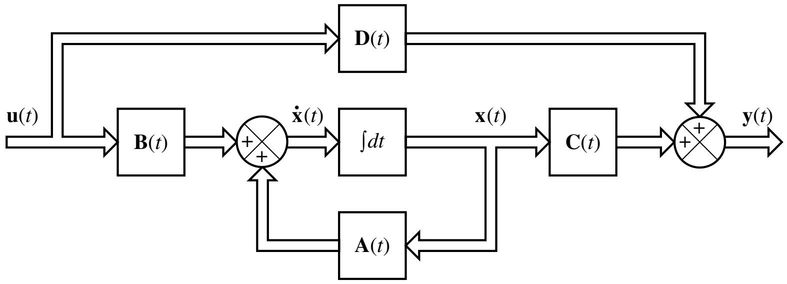


**Figure 2–14** Block diagram of the linear, continuous-time control system represented in state space.

the operating state are discussed in Section 2–7.) A block diagram representation of Equations (2–12) and (2–13) is shown in Figure 2–14.

If vector functions **f** and **g** do not involve time *t* explicitly then the system is called a time-invariant system. In this case, Equations (2–12) and (2–13) can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2–14}$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2–15}$$

Equation (2–14) is the state equation of the linear, time-invariant system and Equation (2–15) is the output equation for the same system. In this book we shall be concerned mostly with systems described by Equations (2–14) and (2–15).

In what follows we shall present an example for deriving a state equation and output equation.

**EXAMPLE 2–2** Consider the mechanical system shown in Figure 2–15. We assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \tag{2–16}$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2 \tag{2–17}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{2–18}$$

The output equation is

$$y = x_1 \tag{2–19}$$

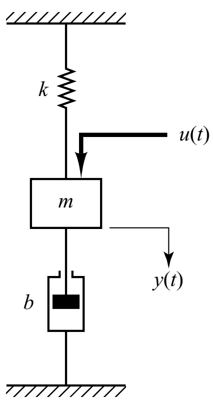


**Figure 2–15** Mechanical system.

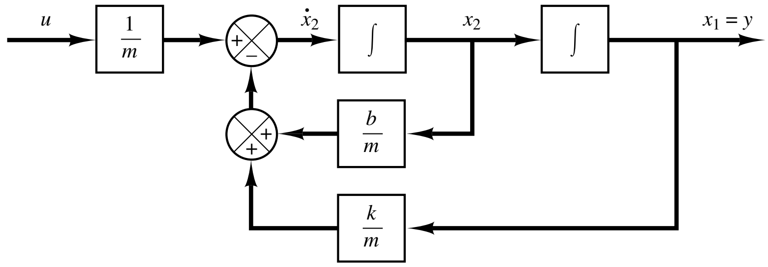


**Figure 2–16**
Block diagram of the mechanical system shown in Figure 2–15.

In a vector-matrix form, Equations (2–17) and (2–18) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{2–20}$$

The output equation, Equation (2–19), can be written as

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2–21}$$

Equation (2–20) is a state equation and Equation (2–21) is an output equation for the system. They are in the standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$

Figure 2–16 is a block diagram for the system. Notice that the outputs of the integrators are state variables.

**Correlation Between Transfer Functions and State-Space Equations.** In what follows we shall show how to derive the transfer function of a single-input, single-output system from the state-space equations.

Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s) \tag{2–22}$$

This system may be represented in state space by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{2–23}$$

$$y = \mathbf{C}\mathbf{x} + Du \tag{2–24}$$

where $\mathbf{x}$ is the state vector, $u$ is the input, and $y$ is the output. The Laplace transforms of Equations (2–23) and (2–24) are given by

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{2–25}$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s) \tag{2–26}$$

Since the transfer function was previously defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions were zero, we set $\mathbf{x}(0)$ in Equation (2–25) to be zero. Then we have

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

By premultiplying $(s\mathbf{I} - \mathbf{A})^{-1}$ to both sides of this last equation, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) \tag{2–27}$$

By substituting Equation (2–27) into Equation (2–26), we get

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s) \tag{2–28}$$

Upon comparing Equation (2–28) with Equation (2–22), we see that

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \tag{2–29}$$

This is the transfer-function expression of the system in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $D$.

Note that the right-hand side of Equation (2–29) involves $(s\mathbf{I} - \mathbf{A})^{-1}$. Hence $G(s)$ can be written as

$$G(s) = \frac{Q(s)}{|s\mathbf{I} - \mathbf{A}|}$$

where $Q(s)$ is a polynomial in $s$. Notice that $|s\mathbf{I} - \mathbf{A}|$ is equal to the characteristic polynomial of $G(s)$. In other words, the eigenvalues of $\mathbf{A}$ are identical to the poles of $G(s)$.

**EXAMPLE 2–3** Consider again the mechanical system shown in Figure 2–15. State-space equations for the system are given by Equations (2–20) and (2–21). We shall obtain the transfer function for the system from the state-space equations.

By substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $D$ into Equation (2–29), we obtain

$$\begin{aligned}
G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\
&= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} + 0 \\
&= [1 \quad 0]\begin{bmatrix} s & -1 \\ \dfrac{k}{m} & s + \dfrac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}
\end{aligned}$$

Note that

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

(Refer to Appendix C for the inverse of the $2 \times 2$ matrix.)
Thus, we have

$$\begin{aligned} G(s) &= [1 \quad 0] \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &= \frac{1}{ms^2 + bs + k} \end{aligned}$$

which is the transfer function of the system. The same transfer function can be obtained from Equation (2–16).

**Transfer Matrix.** Next, consider a multiple-input, multiple-output system. Assume that there are $r$ inputs $u_1, u_2, \ldots, u_r$, and $m$ outputs $y_1, y_2, \ldots, y_m$. Define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

The transfer matrix $\mathbf{G}(s)$ relates the output $\mathbf{Y}(s)$ to the input $\mathbf{U}(s)$, or

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

where $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

[The derivation for this equation is the same as that for Equation (2–29).] Since the input vector $\mathbf{u}$ is $r$ dimensional and the output vector $\mathbf{y}$ is $m$ dimensional, the transfer matrix $\mathbf{G}$(s) is an $m \times r$ matrix.

## 2–5 STATE-SPACE REPRESENTATION OF SCALAR DIFFERENTIAL EQUATION SYSTEMS

A dynamic system consisting of a finite number of lumped elements may be described by ordinary differential equations in which time is the independent variable. By use of vector-matrix notation, an $n$th-order differential equation may be expressed by a first-order vector-matrix differential equation. If $n$ elements of the vector are a set of state variables, then the vector-matrix differential equation is a *state* equation. In this section we shall present methods for obtaining state-space representations of continuous-time systems.

**State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms.** Consider the following *n*th-order system:

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{2–30}$$

Noting that the knowledge of $y(0), \dot{y}(0), \ldots, \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of $n$ state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher-order derivative terms are inaccurate, due to the noise effects inherent in any practical situations, such a choice of the state variables may not be desirable.)

Let us define

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ x_n &= \overset{(n-1)}{y} \end{aligned}$$

Then Equation (2–30) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u \end{aligned}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{2–31}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \tag{2–32}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

[Note that $D$ in Equation (2–24) is zero.] The first-order differential equation, Equation (2–31), is the state equation, and the algebraic equation, Equation (2–32), is the output equation.

Note that the state-space representation for the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by Equations (2–31) and (2–32).

**State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms.** Consider the differential equation system that involves derivatives of the forcing function, such as

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \tag{2–33}$$

The main problem in defining the state variables for this case lies in the derivative terms of the input $u$. The state variables must be such that they will eliminate the derivatives of $u$ in the state equation.

One way to obtain a state equation and output equation for this case is to define the following $n$ variables as a set of $n$ state variables:

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 &= \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= \overset{(n-1)}{y} - \beta_0\overset{(n-1)}{u} - \beta_1\overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u \end{aligned} \tag{2–34}$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are determined from

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\cdot \\
&\cdot \\
&\cdot \\
\beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0
\end{aligned} \tag{2–35}$$

With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables.) With the present choice of state variables, we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned} \tag{2–36}$$

where $\beta_n$ is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

[To derive Equation (2–36), see Problem **A–2–6**.] In terms of vector-matrix equations, Equation (2–36) and the output equation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{2–37}$$

$$y = \mathbf{C}\mathbf{x} + Du \tag{2–38}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \qquad D = \beta_0 = b_0$$

In this state-space representation, matrices **A** and **C** are exactly the same as those for the system of Equation (2–30). The derivatives on the right-hand side of Equation (2–33) affect only the elements of the **B** matrix.

Note that the state-space representation for the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by Equations (2–37) and (2–38).

There are many ways to obtain state-space representations of systems. Methods for obtaining canonical representations of systems in state space (such as controllable canonical form, observable canonical form, diagonal canonical form, and Jordan canonical form) are presented in Chapter 9.

MATLAB can also be used to obtain state-space representations of systems from transfer-function representations, and vice versa. This subject is presented in Section 2–6.

## 2–6 TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful to transform the system model from transfer function to state space, and vice versa. We shall begin our discussion with transformation from transfer function to state space.

Let us write the closed-loop transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

```
[A,B,C,D] = tf2ss(num,den)
```

will give a state-space representation. It is important to note that the state-space representation for any system is not unique. There are many (infinitely many) state-space representations for the same system. The MATLAB command gives one possible such state-space representation.

**Transformation from Transfer Function to State Space Representation.** Consider the transfer-function system

$$\frac{Y(s)}{U(s)} = \frac{s}{(s + 10)(s^2 + 4s + 16)}$$

$$= \frac{s}{s^3 + 14s^2 + 56s + 160} \tag{2–39}$$

There are many (infinitely many) possible state-space representations for this system. One possible state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Another possible state-space representation (among infinitely many alternatives) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{2–40}$$

$$y = [0 \quad 1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \tag{2–41}$$

MATLAB transforms the transfer function given by Equation (2–39) into the state-space representation given by Equations (2–40) and (2–41). For the example system considered here, MATLAB Program 2–2 will produce matrices **A**, **B**, **C**, and *D*.

**MATLAB Program 2–2**

```
num = [1    0];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =

   -14   -56  -160
     1     0     0
     0     1     0

B =

     1
     0
     0

C =

     0     1     0

D =

     0
```

**Transformation from State Space Representation to Transfer Function.** To obtain the transfer function from state-space equations, use the following command:

```
[num,den] = ss2tf(A,B,C,D,iu)
```

iu must be specified for systems with more than one input. For example, if the system has three inputs ($u1$, $u2$, $u3$), then iu must be either 1, 2, or 3, where 1 implies $u1$, 2 implies $u2$, and 3 implies $u3$.

If the system has only one input, then either

```
[num,den] = ss2tf(A,B,C,D)
```

or

$$[\text{num,den}] = \text{ss2tf}(A,B,C,D,1)$$

may be used. For the case where the system has multiple inputs and multiple outputs, see Problem **A–2–12**.

**EXAMPLE 2–4** Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATLAB Program 2-3 will produce the transfer function for the given system. The transfer function obtained is given by

$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

**MATLAB Program 2–3**

```
A = [0   1   0;   0   0   1;   -5   -25   -5];
B = [0; 25; -120];
C = [1   0     0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

   0   0.0000   25.0000  5.0000

den

   1.0000   5.0000   25.0000  5.0000

% ***** The same result can be obtained by entering the following command: *****

[num,den] = ss2tf(A,B,C,D,1)

num =

   0   0.0000   25.0000  5.0000

den =

   1.0000   5.0000   25.0000  5.0000
```

## 2–7 LINEARIZATION OF NONLINEAR MATHEMATICAL MODELS

**Nonlinear Systems.** A system is nonlinear if the principle of superposition does not apply. Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results.

Although many physical relationships are often represented by linear equations, in most cases actual relationships are not quite linear. In fact, a careful study of physical systems reveals that even so-called "linear systems" are really linear only in limited operating ranges. In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve nonlinear relationships among the variables. For example, the output of a component may saturate for large input signals. There may be a dead space that affects small signals. (The dead space of a component is a small range of input variations to which the component is insensitive.) Square-law nonlinearity may occur in some components. For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity.

**Linearization of Nonlinear Systems.** In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around the equilibrium. (It should be pointed out that there are many exceptions to such a case.) However, if the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system. Such a linear system is equivalent to the nonlinear system considered within a limited operating range. Such a linearized model (linear, time-invariant model) is very important in control engineering.

The linearization procedure to be presented in the following is based on the expansion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term. Because we neglect higher-order terms of the Taylor series expansion, these neglected terms must be small enough; that is, the variables deviate only slightly from the operating condition. (Otherwise, the result will be inaccurate.)

**Linear Approximation of Nonlinear Mathematical Models.** To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition. Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x) \tag{2–42}$$

If the normal operating condition corresponds to $\bar{x}, \bar{y}$, then Equation (2–42) may be expanded into a Taylor series about this point as follows:

$$\begin{aligned} y &= f(x) \\ &= f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x - \bar{x})^2 + \cdots \end{aligned} \tag{2–43}$$

where the derivatives $df/dx, d^2f/dx^2, \ldots$ are evaluated at $x = \bar{x}$. If the variation $x - \bar{x}$ is small, we may neglect the higher-order terms in $x - \bar{x}$. Then Equation (2–43) may be written as

$$y = \bar{y} + K(x - \bar{x}) \tag{2–44}$$

where

$$\bar{y} = f(\bar{x})$$

$$K = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

Equation (2–44) may be rewritten as

$$y - \bar{y} = K(x - \bar{x}) \tag{2–45}$$

which indicates that $y - \bar{y}$ is proportional to $x - \bar{x}$. Equation (2–45) gives a linear mathematical model for the nonlinear system given by Equation (2–42) near the operating point $x = \bar{x}, y = \bar{y}$.

Next, consider a nonlinear system whose output $y$ is a function of two inputs $x_1$ and $x_2$, so that

$$y = f(x_1, x_2) \tag{2–46}$$

To obtain a linear approximation to this nonlinear system, we may expand Equation (2–46) into a Taylor series about the normal operating point $\bar{x}_1, \bar{x}_2$. Then Equation (2–46) becomes

$$\begin{aligned} y = f(\bar{x}_1, \bar{x}_2) &+ \left[\frac{\partial f}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f}{\partial x_2}(x_2 - \bar{x}_2)\right] \\ &+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - \bar{x}_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) \right. \\ &\left. + \frac{\partial^2 f}{\partial x_2^2}(x_2 - \bar{x}_2)^2\right] + \cdots \end{aligned}$$

where the partial derivatives are evaluated at $x_1 = \bar{x}_1, x_2 = \bar{x}_2$. Near the normal operating point, the higher-order terms may be neglected. The linear mathematical model of this nonlinear system in the neighborhood of the normal operating condition is then given by

$$y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_2)$$

where

$$\bar{y} = f(\bar{x}_1, \bar{x}_2)$$

$$K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2}$$

$$K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2}$$

The linearization technique presented here is valid in the vicinity of the operating condition. If the operating conditions vary widely, however, such linearized equations are not adequate, and nonlinear equations must be dealt with. It is important to remember that a particular mathematical model used in analysis and design may accurately represent the dynamics of an actual system for certain operating conditions, but may not be accurate for other operating conditions.

**EXAMPLE 2–5** Linearize the nonlinear equation

$$z = xy$$

in the region $5 \le x \le 7$, $10 \le y \le 12$. Find the error if the linearized equation is used to calculate the value of $z$ when $x = 5$, $y = 10$.

Since the region considered is given by $5 \le x \le 7$, $10 \le y \le 12$, choose $\bar{x} = 6$, $\bar{y} = 11$. Then $\bar{z} = \bar{x}\bar{y} = 66$. Let us obtain a linearized equation for the nonlinear equation near a point $\bar{x} = 6$, $\bar{y} = 11$.

Expanding the nonlinear equation into a Taylor series about point $x = \bar{x}$, $y = \bar{y}$ and neglecting the higher-order terms, we have

$$z - \bar{z} = a(x - \bar{x}) + b(y - \bar{y})$$

where

$$a = \left.\frac{\partial(xy)}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{y} = 11$$

$$b = \left.\frac{\partial(xy)}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{x} = 6$$

Hence the linearized equation is

$$z - 66 = 11(x - 6) + 6(y - 11)$$

or

$$z = 11x + 6y - 66$$

When $x = 5$, $y = 10$, the value of $z$ given by the linearized equation is

$$z = 11x + 6y - 66 = 55 + 60 - 66 = 49$$

The exact value of $z$ is $z = xy = 50$. The error is thus $50 - 49 = 1$. In terms of percentage, the error is 2%.

## EXAMPLE PROBLEMS AND SOLUTIONS

**A–2–1.** Simplify the block diagram shown in Figure 2–17.

**Solution.** First, move the branch point of the path involving $H_1$ outside the loop involving $H_2$, as shown in Figure 2–18(a). Then eliminating two loops results in Figure 2–18(b). Combining two blocks into one gives Figure 2–18(c).

**A–2–2.** Simplify the block diagram shown in Figure 2–19. Obtain the transfer function relating $C(s)$ and $R(s)$.

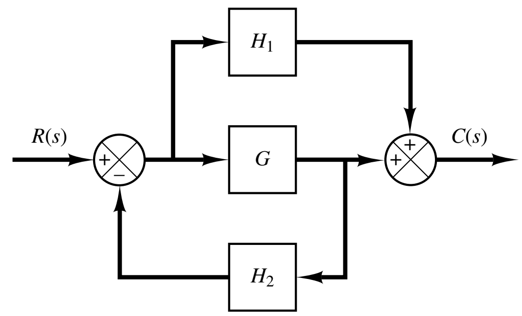


**Figure 2–17** Block diagram of a system.

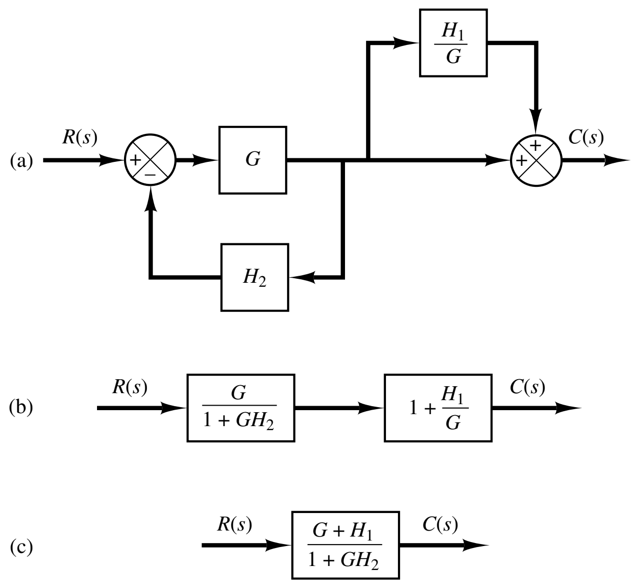


**Figure 2–18** Simplified block diagrams for the system shown in Figure 2–17.

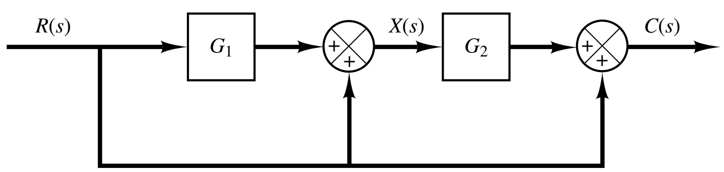


**Figure 2–19** Block diagram of a system.

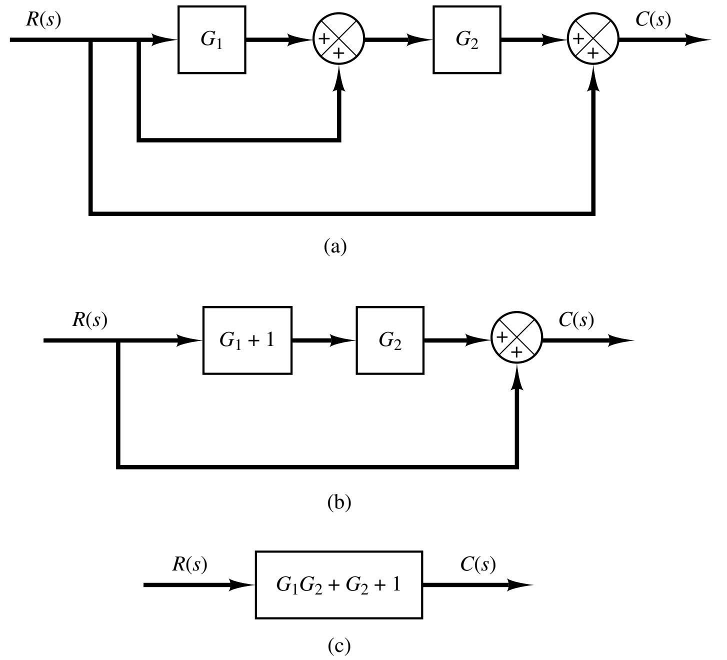


**Figure 2–20** Reduction of the block diagram shown in Figure 2–19.

**Solution.** The block diagram of Figure 2–19 can be modified to that shown in Figure 2–20(a). Eliminating the minor feedforward path, we obtain Figure 2–20(b), which can be simplified to Figure 2–20(c). The transfer function $C(s)/R(s)$ is thus given by

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

The same result can also be obtained by proceeding as follows: Since signal $X(s)$ is the sum of two signals $G_1R(s)$ and $R(s)$, we have

$$X(s) = G_1R(s) + R(s)$$

The output signal $C(s)$ is the sum of $G_2X(s)$ and $R(s)$. Hence

$$C(s) = G_2X(s) + R(s) = G_2\left[G_1R(s) + R(s)\right] + R(s)$$

And so we have the same result as before:

$$\frac{C(s)}{R(s)} = G_1G_2 + G_2 + 1$$

**A–2–3.** Simplify the block diagram shown in Figure 2–21. Then obtain the closed-loop transfer function $C(s)/R(s)$.

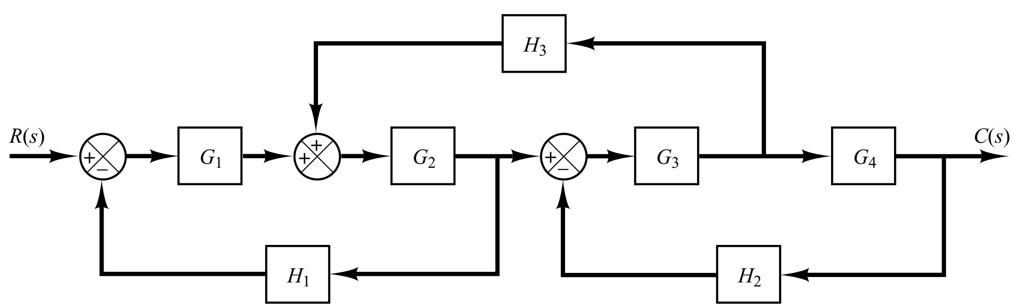


**Figure 2–21** Block diagram of a system.

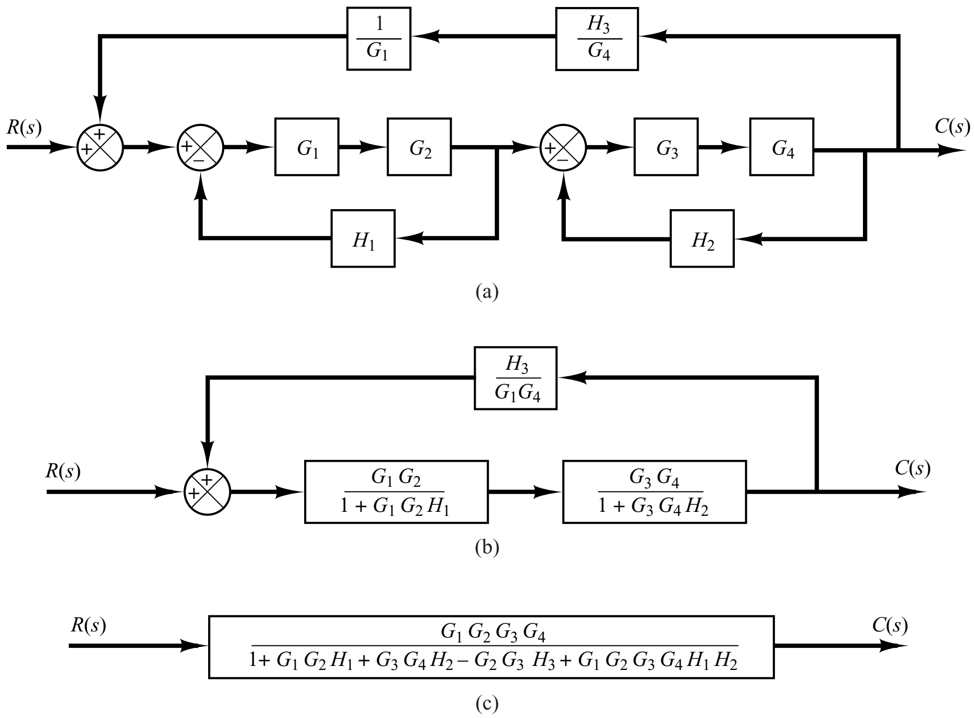


**Figure 2–22** Successive reductions of the block diagram shown in Figure 2–21.

**Solution.** First move the branch point between $G_3$ and $G_4$ to the right-hand side of the loop containing $G_3$, $G_4$, and $H_2$. Then move the summing point between $G_1$ and $G_2$ to the left-hand side of the first summing point. See Figure 2–22(a). By simplifying each loop, the block diagram can be modified as shown in Figure 2–22(b). Further simplification results in Figure 2–22(c), from which the closed-loop transfer function $C(s)/R(s)$ is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2H_1 + G_3G_4H_2 - G_2G_3H_3 + G_1G_2G_3G_4H_1H_2}$$

**A–2–4.** Obtain transfer functions $C(s)/R(s)$ and $C(s)/D(s)$ of the system shown in Figure 2–23.

**Solution.** From Figure 2–23 we have

$$U(s) = G_f R(s) + G_c E(s) \qquad (2\text{–}47)$$

$$C(s) = G_p\left[D(s) + G_1 U(s)\right] \qquad (2\text{–}48)$$

$$E(s) = R(s) - HC(s) \qquad (2\text{–}49)$$

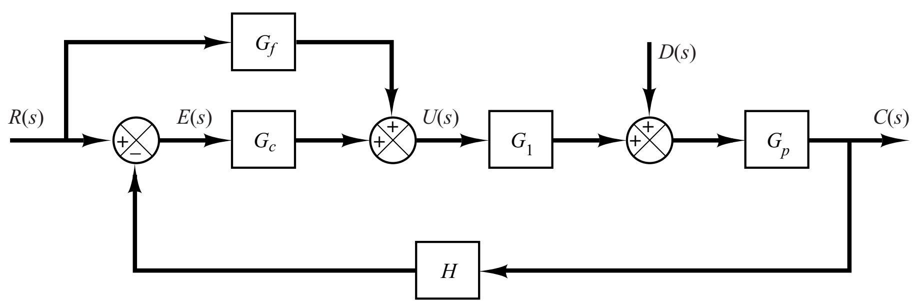


**Figure 2–23** Control system with reference input and disturbance input.

By substituting Equation (2–47) into Equation (2–48), we get

$$C(s) = G_p D(s) + G_1 G_p \left[ G_f R(s) + G_c E(s) \right] \tag{2–50}$$

By substituting Equation (2–49) into Equation (2–50), we obtain

$$C(s) = G_p D(s) + G_1 G_p \{ G_f R(s) + G_c [R(s) - HC(s)] \}$$

Solving this last equation for $C(s)$, we get

$$C(s) + G_1 G_p G_c H C(s) = G_p D(s) + G_1 G_p (G_f + G_c) R(s)$$

Hence

$$C(s) = \frac{G_p D(s) + G_1 G_p (G_f + G_c) R(s)}{1 + G_1 G_p G_c H} \tag{2–51}$$

Note that Equation (2–51) gives the response $C(s)$ when both reference input $R(s)$ and disturbance input $D(s)$ are present.

To find transfer function $C(s)/R(s)$, we let $D(s) = 0$ in Equation (2–51). Then we obtain

$$\frac{C(s)}{R(s)} = \frac{G_1 G_p (G_f + G_c)}{1 + G_1 G_p G_c H}$$

Similarly, to obtain transfer function $C(s)/D(s)$, we let $R(s) = 0$ in Equation (2–51). Then $C(s)/D(s)$ can be given by

$$\frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1 G_p G_c H}$$

**A–2–5.** Figure 2–24 shows a system with two inputs and two outputs. Derive $C_1(s)/R_1(s)$, $C_1(s)/R_2(s)$, $C_2(s)/R_1(s)$, and $C_2(s)/R_2(s)$. (In deriving outputs for $R_1(s)$, assume that $R_2(s)$ is zero, and vice versa.)

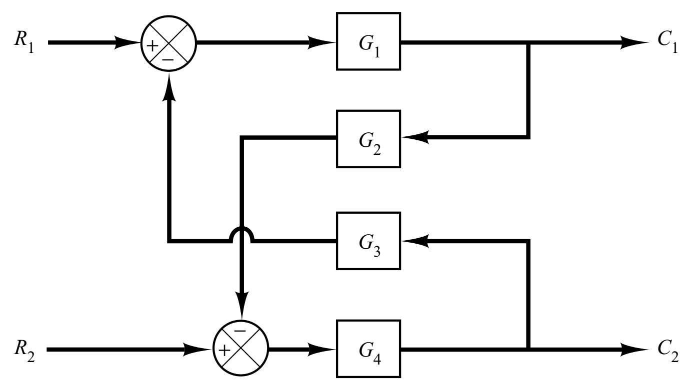


**Figure 2–24** System with two inputs and two outputs.

**Solution.** From the figure, we obtain

$$C_1 = G_1(R_1 - G_3C_2) \tag{2–52}$$

$$C_2 = G_4(R_2 - G_2C_1) \tag{2–53}$$

By substituting Equation (2–53) into Equation (2–52), we obtain

$$C_1 = G_1[R_1 - G_3G_4(R_2 - G_2C_1)] \tag{2–54}$$

By substituting Equation (2–52) into Equation (2–53), we get

$$C_2 = G_4[R_2 - G_2G_1(R_1 - G_3C_2)] \tag{2–55}$$

Solving Equation (2–54) for $C_1$, we obtain

$$C_1 = \frac{G_1R_1 - G_1G_3G_4R_2}{1 - G_1G_2G_3G_4} \tag{2–56}$$

Solving Equation (2–55) for $C_2$ gives

$$C_2 = \frac{-G_1G_2G_4R_1 + G_4R_2}{1 - G_1G_2G_3G_4} \tag{2–57}$$

Equations (2–56) and (2–57) can be combined in the form of the transfer matrix as follows:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \dfrac{G_1}{1 - G_1G_2G_3G_4} & -\dfrac{G_1G_3G_4}{1 - G_1G_2G_3G_4} \\ -\dfrac{G_1G_2G_4}{1 - G_1G_2G_3G_4} & \dfrac{G_4}{1 - G_1G_2G_3G_4} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

Then the transfer functions $C_1(s)/R_1(s)$, $C_1(s)/R_2(s)$, $C_2(s)/R_1(s)$ and $C_2(s)/R_2(s)$ can be obtained as follows:

$$\frac{C_1(s)}{R_1(s)} = \frac{G_1}{1 - G_1G_2G_3G_4}, \qquad \frac{C_1(s)}{R_2(s)} = -\frac{G_1G_3G_4}{1 - G_1G_2G_3G_4}$$

$$\frac{C_2(s)}{R_1(s)} = -\frac{G_1G_2G_4}{1 - G_1G_2G_3G_4}, \qquad \frac{C_2(s)}{R_2(s)} = \frac{G_4}{1 - G_1G_2G_3G_4}$$

Note that Equations (2–56) and (2–57) give responses $C_1$ and $C_2$, respectively, when both inputs $R_1$ and $R_2$ are present.

Notice that when $R_2(s) = 0$, the original block diagram can be simplified to those shown in Figures 2–25(a) and (b). Similarly, when $R_1(s) = 0$, the original block diagram can be simplified to those shown in Figures 2–25(c) and (d). From these simplified block diagrams we can also obtain $C_1(s)/R_1(s)$, $C_2(s)/R_1(s)$, $C_1(s)/R_2(s)$, and $C_2(s)/R_2(s)$, as shown to the right of each corresponding block diagram.

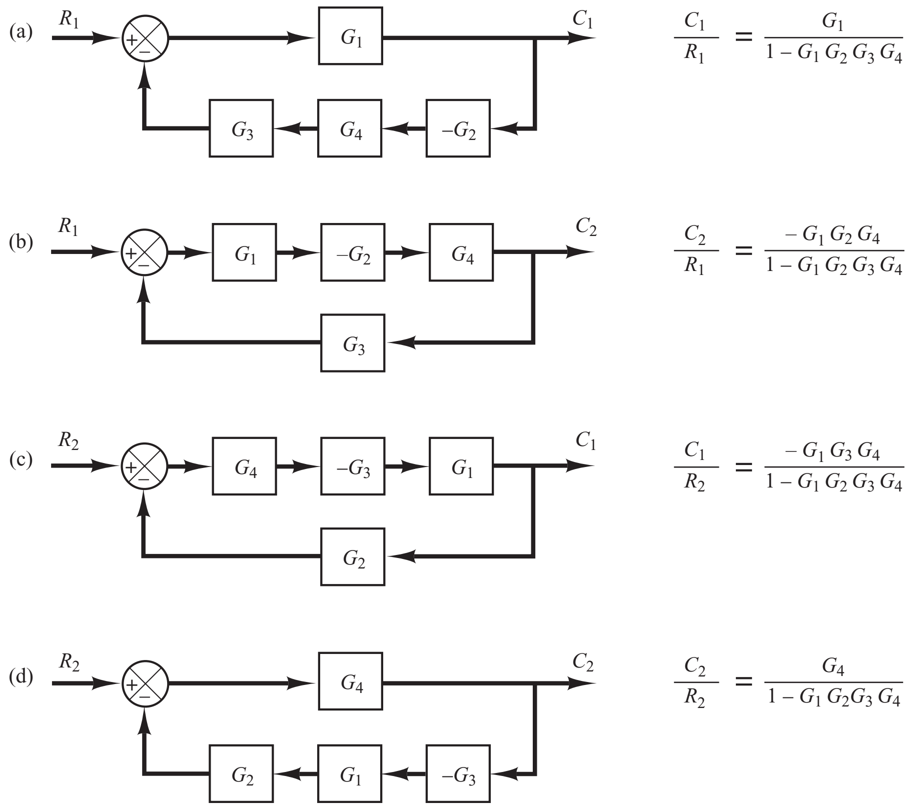


**Figure 2–25** Simplified block diagrams and corresponding closed-loop transfer functions.

**A–2–6.** Show that for the differential equation system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u \tag{2–58}$$

state and output equations can be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u \tag{2–59}$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u \tag{2–60}$$

where state variables are defined by

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

and

$$\beta_0 = b_0$$
$$\beta_1 = b_1 - a_1\beta_0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$
$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

**Solution.** From the definition of state variables $x_2$ and $x_3$, we have

$$\dot{x}_1 = x_2 + \beta_1 u \tag{2–61}$$
$$\dot{x}_2 = x_3 + \beta_2 u \tag{2–62}$$

To derive the equation for $\dot{x}_3$, we first note from Equation (2–58) that

$$\dddot{y} = -a_1\ddot{y} - a_2\dot{y} - a_3 y + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Since

$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u$$

we have

$$\begin{aligned}
\dot{x}_3 &= \dddot{y} - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= (-a_1\ddot{y} - a_2\dot{y} - a_3 y) + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= -a_1(\ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u) - a_1\beta_0\ddot{u} - a_1\beta_1\dot{u} - a_1\beta_2 u \\
&\quad - a_2(\dot{y} - \beta_0\dot{u} - \beta_1 u) - a_2\beta_0\dot{u} - a_2\beta_1 u - a_3(y - \beta_0 u) - a_3\beta_0 u \\
&\quad + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_0 - \beta_0)\dddot{u} + (b_1 - \beta_1 - a_1\beta_0)\ddot{u} \\
&\quad + (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)\dot{u} + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u
\end{aligned}$$

Hence, we get

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u \tag{2–63}$$

Combining Equations (2–61), (2–62), and (2–63) into a vector-matrix equation, we obtain Equation (2–59). Also, from the definition of state variable $x_1$, we get the output equation given by Equation (2–60).

**A–2–7.** Obtain a state-space equation and output equation for the system defined by

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s + 2}{s^3 + 4s^2 + 5s + 2}$$

**Solution.** From the given transfer function, the differential equation for the system is

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\dddot{u} + \ddot{u} + \dot{u} + 2u$$

Comparing this equation with the standard equation given by Equation (2–33), rewritten

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we find

$$a_1 = 4, \quad a_2 = 5, \quad a_3 = 2$$
$$b_0 = 2, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 2$$

Referring to Equation (2–35), we have

$$\beta_0 = b_0 = 2$$
$$\beta_1 = b_1 - a_1\beta_0 = 1 - 4 \times 2 = -7$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 1 - 4 \times (-7) - 5 \times 2 = 19$$
$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$
$$= 2 - 4 \times 19 - 5 \times (-7) - 2 \times 2 = -43$$

Referring to Equation (2–34), we define

$$x_1 = y - \beta_0 u = y - 2u$$
$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 + 7u$$
$$x_3 = \dot{x}_2 - \beta_2 u = \dot{x}_2 - 19u$$

Then referring to Equation (2–36),

$$\dot{x}_1 = x_2 - 7u$$
$$\dot{x}_2 = x_3 + 19u$$
$$\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + \beta_3 u$$
$$= -2x_1 - 5x_2 - 4x_3 - 43u$$

Hence, the state-space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u$$

This is one possible state-space representation of the system. There are many (infinitely many) others. If we use MATLAB, it produces the following state-space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [-7 \quad -9 \quad -2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u$$

See MATLAB Program 2-4. (Note that all state-space representations for the same system are equivalent.)

**MATLAB Program 2–4**

```
num = [2  1  1  2];
den = [1  4  5  2];
[A,B,C,D] = tf2ss(num,den)

A =

    -4   -5   -2
     1    0    0
     0    1    0

B =

     1
     0
     0

C =

    -7   -9   -2

D =

     2
```

**A–2–8.** Obtain a state-space model of the system shown in Figure 2–26.

**Solution.** The system involves one integrator and two delayed integrators. The output of each integrator or delayed integrator can be a state variable. Let us define the output of the plant as $x_1$, the output of the controller as $x_2$, and the output of the sensor as $x_3$. Then we obtain

$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s + 5}$$

$$\frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s}$$

$$\frac{X_3(s)}{X_1(s)} = \frac{1}{s + 1}$$

$$Y(s) = X_1(s)$$

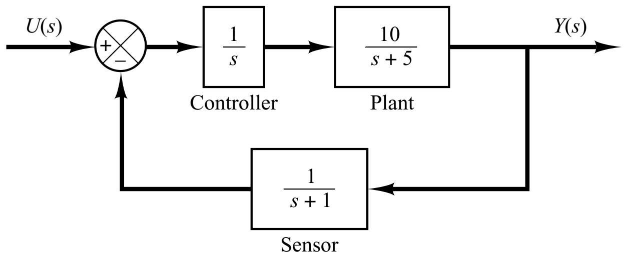


**Figure 2–26**
Control system.

which can be rewritten as

$$
\begin{aligned}
sX_1(s) &= -5X_1(s) + 10X_2(s) \\
sX_2(s) &= -X_3(s) + U(s) \\
sX_3(s) &= X_1(s) - X_3(s) \\
Y(s) &= X_1(s)
\end{aligned}
$$

By taking the inverse Laplace transforms of the preceding four equations, we obtain

$$
\begin{aligned}
\dot{x}_1 &= -5x_1 + 10x_2 \\
\dot{x}_2 &= -x_3 + u \\
\dot{x}_3 &= x_1 - x_3 \\
y &= x_1
\end{aligned}
$$

Thus, a state-space model of the system in the standard form is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

It is important to note that this is not the only state-space representation of the system. Infinitely many other state-space representations are possible. However, the number of state variables is the same in any state-space representation of the same system. In the present system, the number of state variables is three, regardless of what variables are chosen as state variables.

**A–2–9.** Obtain a state-space model for the system shown in Figure 2–27(a).

**Solution.** First, notice that $(as + b)/s^2$ involves a derivative term. Such a derivative term may be avoided if we modify $(as + b)/s^2$ as

$$
\frac{as + b}{s^2} = \left(a + \frac{b}{s}\right)\frac{1}{s}
$$

Using this modification, the block diagram of Figure 2–27(a) can be modified to that shown in Figure 2–27(b).

Define the outputs of the integrators as state variables, as shown in Figure 2–27(b). Then from Figure 2–27(b) we obtain

$$
\begin{aligned}
\frac{X_1(s)}{X_2(s) + a\left[U(s) - X_1(s)\right]} &= \frac{1}{s} \\
\frac{X_2(s)}{U(s) - X_1(s)} &= \frac{b}{s} \\
Y(s) &= X_1(s)
\end{aligned}
$$

which may be modified to

$$
\begin{aligned}
sX_1(s) &= X_2(s) + a\left[U(s) - X_1(s)\right] \\
sX_2(s) &= -bX_1(s) + bU(s) \\
Y(s) &= X_1(s)
\end{aligned}
$$

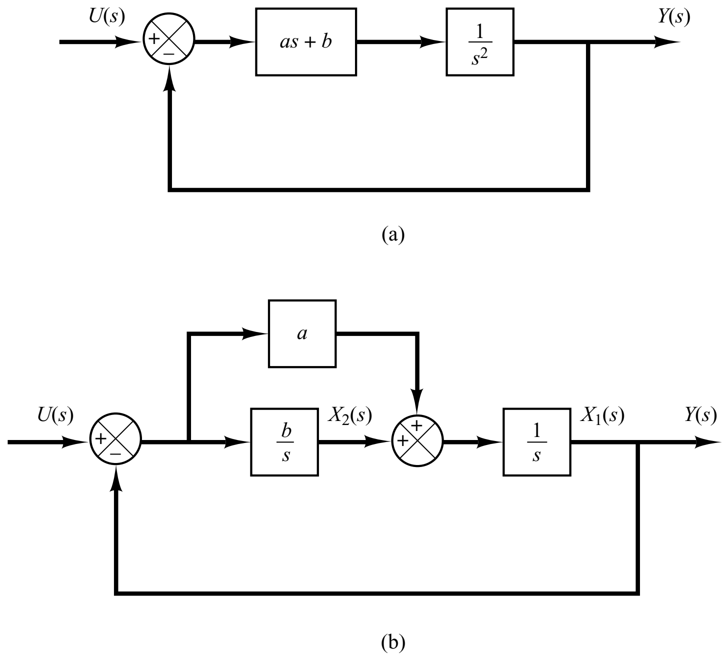


**Figure 2–27**
(a) Control system;
(b) modified block diagram.

Taking the inverse Laplace transforms of the preceding three equations, we obtain

$$\dot{x}_1 = -ax_1 + x_2 + au$$
$$\dot{x}_2 = -bx_1 + bu$$
$$y = x_1$$

Rewriting the state and output equations in the standard vector-matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**A–2–10.** Obtain a state-space representation of the system shown in Figure 2–28(a).

**Solution.** In this problem, first expand $(s + z)/(s + p)$ into partial fractions.

$$\frac{s + z}{s + p} = 1 + \frac{z - p}{s + p}$$

Next, convert $K/[s(s + a)]$ into the product of $K/s$ and $1/(s + a)$. Then redraw the block diagram, as shown in Figure 2–28(b). Defining a set of state variables, as shown in Figure 2–28(b), we obtain the following equations:

$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = -Kx_1 + Kx_3 + Ku$$
$$\dot{x}_3 = -(z - p)x_1 - px_3 + (z - p)u$$
$$y = x_1$$

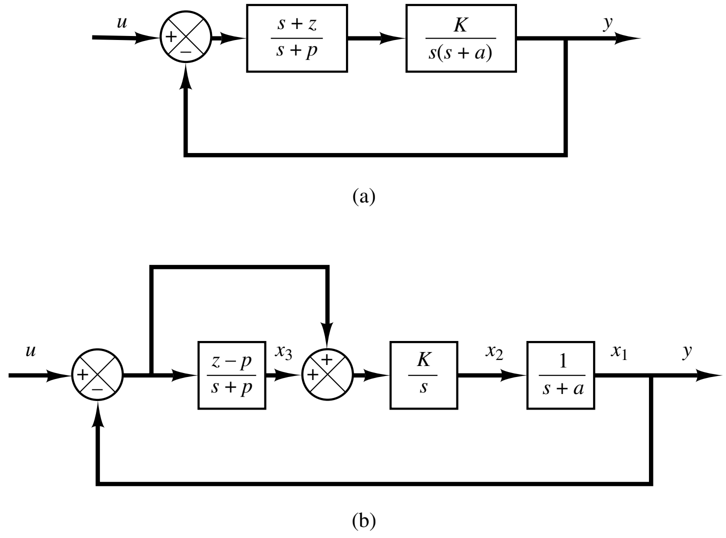


**Figure 2–28**
(a) Control system; (b) block diagram defining state variables for the system.

Rewriting gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a & 1 & 0 \\ -K & 0 & K \\ -(z-p) & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ z-p \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that the output of the integrator and the outputs of the first-order delayed integrators $[1/(s+a)$ and $(z-p)/(s+p)]$ are chosen as state variables. It is important to remember that the output of the block $(s+z)/(s+p)$ in Figure 2–28(a) cannot be a state variable, because this block involves a derivative term, $s+z$.

**A–2–11.** Obtain the transfer function of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**Solution.** Referring to Equation (2–29), the transfer function $G(s)$ is given by

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

In this problem, matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $D$ are

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0], \qquad D = 0$$

Hence

$$G(s) = [1 \quad 0 \quad 0]\begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s+2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

**A–2–12.** Consider a system with multiple inputs and multiple outputs. When the system has more than one output, the MATLAB command

```
[NUM,den] = ss2tf(A,B,C,D,iu)
```

produces transfer functions for all outputs to each input. (The numerator coefficients are returned to matrix NUM with as many rows as there are outputs.)

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved: $Y_1(s)/U_1(s)$, $Y_2(s)/U_1(s)$, $Y_1(s)/U_2(s)$, and $Y_2(s)/U_2(s)$. (When considering input $u_1$, we assume that input $u_2$ is zero and vice versa.)

**Solution.** MATLAB Program 2-5 produces four transfer functions.
This is the MATLAB representation of the following four transfer functions:

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2 + 4s + 25}, \qquad \frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2 + 4s + 25}$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2 + 4s + 25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2 + 4s + 25}$$

**MATLAB Program 2–5**

```
A = [0   1;-25   -4];
B = [1   1;0   1];
C = [1   0;0   1];
D = [0   0;0   0];
[NUM,den] = ss2tf(A,B,C,D,1)

NUM =
         0     1     4
         0     0   –25

den =

         1     4    25

[NUM,den] = ss2tf(A,B,C,D,2)

NUM =

      0   1.0000     5.0000
      0   1.0000   -25.0000

den =

         1     4    25
```

**A–2–13.** Linearize the nonlinear equation

$$z = x^2 + 4xy + 6y^2$$

in the region defined by $8 \le x \le 10, 2 \le y \le 4$.

**Solution.** Define

$$f(x, y) = z = x^2 + 4xy + 6y^2$$

Then

$$z = f(x, y) = f(\bar{x}, \bar{y}) + \left[\frac{\partial f}{\partial x}(x - \bar{x}) + \frac{\partial f}{\partial y}(y - \bar{y})\right]_{x=\bar{x},\, y=\bar{y}} + \cdots$$

where we choose $\bar{x} = 9, \bar{y} = 3$.

Since the higher-order terms in the expanded equation are small, neglecting these higher-order terms, we obtain

$$z - \bar{z} = K_1(x - \bar{x}) + K_2(y - \bar{y})$$

where

$$K_1 = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = 2\bar{x} + 4\bar{y} = 2 \times 9 + 4 \times 3 = 30$$

$$K_2 = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = 4\bar{x} + 12\bar{y} = 4 \times 9 + 12 \times 3 = 72$$

$$\bar{z} = \bar{x}^2 + 4\bar{x}\bar{y} + 6\bar{y}^2 = 9^2 + 4 \times 9 \times 3 + 6 \times 9 = 243$$

Thus

$$z - 243 = 30(x - 9) + 72(y - 3)$$

Hence a linear approximation of the given nonlinear equation near the operating point is

$$z - 30x - 72y + 243 = 0$$

## PROBLEMS

**B–2–1.** Simplify the block diagram shown in Figure 2–29 and obtain the closed-loop transfer function $C(s)/R(s)$.

**B–2–2.** Simplify the block diagram shown in Figure 2–30 and obtain the closed-loop transfer function $C(s)/R(s)$.

**B–2–3.** Simplify the block diagram shown in Figure 2–31 and obtain the closed-loop transfer function $C(s)/R(s)$.

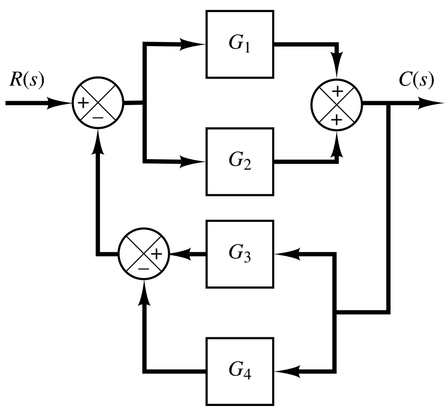


**Figure 2–29**
Block diagram of a system.

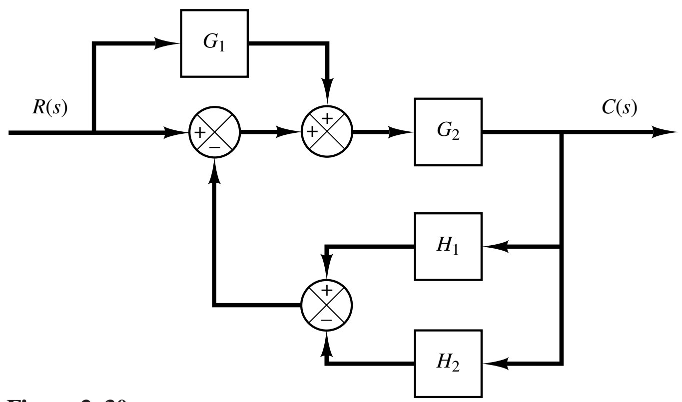


**Figure 2–30**
Block diagram of a system.

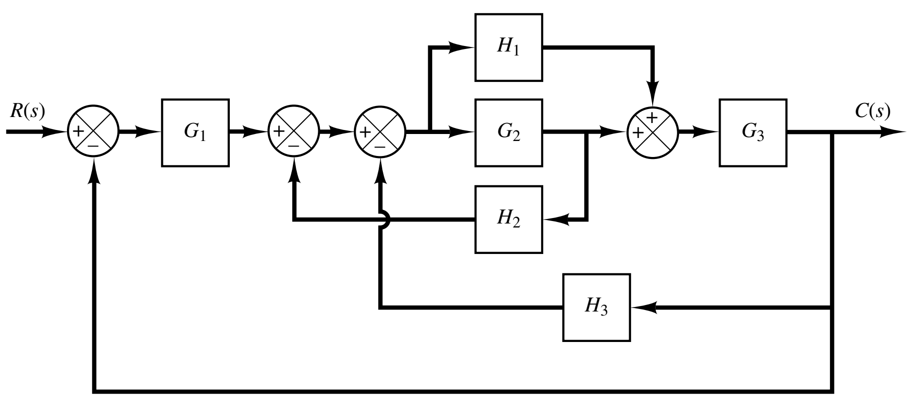


**Figure 2–31**
Block diagram of a system.

**B–2–4.** Consider industrial automatic controllers whose control actions are proportional, integral, proportional-plus-integral, proportional-plus-derivative, and proportional-plus-integral-plus-derivative. The transfer functions of these controllers can be given, respectively, by

$$\frac{U(s)}{E(s)} = K_p$$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

$$\frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

where $U(s)$ is the Laplace transform of $u(t)$, the controller output, and $E(s)$ the Laplace transform of $e(t)$, the actuating error signal. Sketch $u(t)$-versus-$t$ curves for each of the five types of controllers when the actuating error signal is

**(a)** $e(t) =$ unit-step function

**(b)** $e(t) =$ unit-ramp function

In sketching curves, assume that the numerical values of $K_p$, $K_i$, $T_i$, and $T_d$ are given as

$$K_p = \text{proportional gain} = 4$$
$$K_i = \text{integral gain} = 2$$
$$T_i = \text{integral time} = 2 \text{ sec}$$
$$T_d = \text{derivative time} = 0.8 \text{ sec}$$

**B–2–5.** Figure 2–32 shows a closed-loop system with a reference input and disturbance input. Obtain the expression for the output $C(s)$ when both the reference input and disturbance input are present.

**B–2–6.** Consider the system shown in Figure 2–33. Derive the expression for the steady-state error when both the reference input $R(s)$ and disturbance input $D(s)$ are present.

**B–2–7.** Obtain the transfer functions $C(s)/R(s)$ and $C(s)/D(s)$ of the system shown in Figure 2–34.

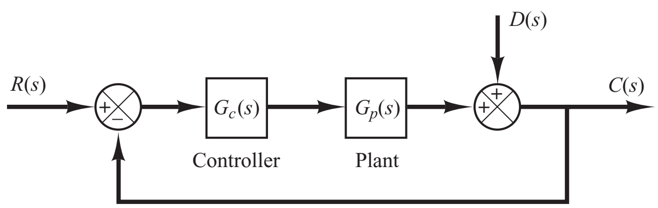


**Figure 2–32**
Closed-loop system.

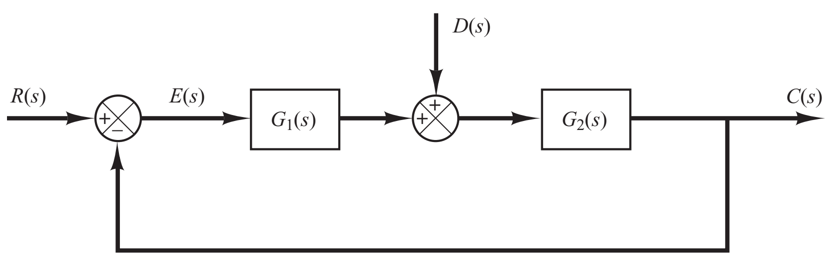


**Figure 2–33**
Control system.

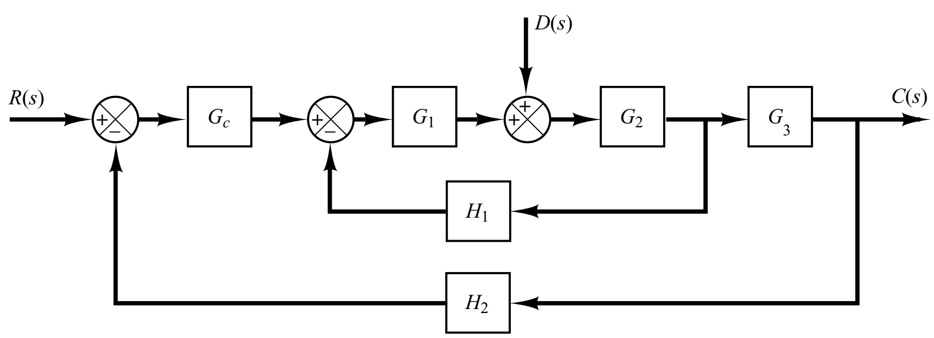


**Figure 2–34**
Control system.

**B–2–8.** Obtain a state-space representation of the system shown in Figure 2–35.

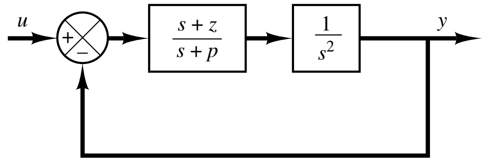


**Figure 2–35**
Control system.

**B–2–9.** Consider the system described by

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = u$$

Derive a state-space representation of the system.

**B–2–10.** Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function of the system.

**B–2–11.** Consider a system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$y = [1 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function $G(s)$ of the system.

**B–2–12.** Obtain the transfer matrix of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**B–2–13.** Linearize the nonlinear equation

$$z = x^2 + 8xy + 3y^2$$

in the region defined by $2 \le x \le 4, 10 \le y \le 12$.

**B–2–14.** Find a linearized equation for

$$y = 0.2x^3$$

about a point $x = 2$.