Q.1 $\int_0^t t \sin t \, dt$

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L(t\sin t) = -\frac{d}{ds} \frac{1}{s^2+1}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$L\int_0^t t\sin t \, dt = \frac{1}{s} \cdot \frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{(s^2+1)^2}$$

Q.2 $\int_0^t\int_0^t\int_0^t t\sin t \, dt\,dt\,dt$

$$\int_0^t\int_0^t\int_0^t t\sin t \, (dt)^3 = \frac{1}{s^3} \cdot \frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{s^2(s^2+1)^2}$$

Q.3 $$\int_0^t \frac{1-e^{3u}}{u}\,du$$

We'll find the laplace transform of

$$\int_0^t \frac{1-e^{3t}}{t}\,dt.$$

$$L(1-e^{3t}) = \frac{1}{s} - \frac{1}{s-3}$$

$$L\left(\frac{1-e^{3t}}{t}\right) = \int_s^\infty \frac{1}{s} - \frac{1}{s-3}$$

$$= \left[\log s - \log(s-3)\right]_s^\infty$$

$$= \left[\log \frac{s}{s-3}\right]_s^\infty$$

$$= \left[\log \frac{1}{1-\frac{3}{s}}\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{1-\frac{3}{s}}\right)$$

$$= 0 - \log \frac{s}{s-3}$$

$$= \log \frac{s-3}{s}$$

$$L\int_0^t \frac{1-e^{3t}}{t}\,dt = \frac{1}{s}\log\frac{s-3}{s}$$

Q.4 $$e^{-4t}\int_0^t u\sin 3u\,du = e^{-4t}\int_0^t t\sin 3t\,dt$$

$$L(\sin 3t) = \frac{3}{s^2+9}$$

$$L(t\sin 3t) = -\frac{d}{ds}\frac{3}{s^2+9}$$

$$= -\frac{(-1)(3)(2s)}{(s^2+9)^2}$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\int_0^t t\sin 3t\,dt = \frac{1}{s}\cdot\frac{6s}{(s^2+9)^2}$$

$$= \frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t\,dt\right) = \frac{6}{((s+4)^2+9)^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

Q.5 $\cosh t\int_0^t e^u \cosh u\,du = \cosh t\int_0^t e^t \cosh t\,dt$

$$L(\cosh t) = \frac{s}{s^2-1}$$

$$L(e^t\cosh t) = \frac{s-1}{(s-1)^2-1}$$

$$L\left(\int_0^t e^t\cosh t\,dt\right) = \frac{1}{s}\cdot\frac{s-1}{(s-1)^2-1}$$

$$L\left(\cosh t\int_0^t e^t\cosh t\,dt\right)$$

$$= L\left(\frac{e^t+e^{-t}}{2}\int_0^t e^t\cosh t\,dt\right)$$

$$= \frac{1}{2}L\left((e^t+e^{-t})\int_0^t e^t\cosh t\,dt\right)$$

$$= \frac{1}{2}L\left(e^t\int_0^t e^t\cosh t\,dt + e^{-t}\int_0^t e^t\cosh t\,dt\right)$$

$$= \frac{1}{2}\left\{L\left(e^t\int_0^t e^t\cosh t\,dt\right) + L\left(e^{-t}\int_0^t e^t\cosh t\,dt\right)\right\}$$

Since $L\left(\int_0^t e^t \cosh t\, dt\right) = \frac{1}{s} \cdot \frac{s-1}{(s-1)^2 - 1}$

$\therefore L\left(e^t \int_0^t e^t \cosh t\, dt\right) = \frac{1}{(s-1)} \cdot \frac{(s-1-1)}{(s-1-1)^2 - 1}$

$$= \frac{(s-2)}{(s-1)\left((s-2)^2 - 1\right)}$$

Also, $L\left(e^{-t} \int_0^t e^t \cosh t\, dt\right) = \frac{1}{s+1} \cdot \frac{(s+1-1)}{(s+1-1)^2 - 1}$

$$= \frac{s}{(s+1)(s^2 - 1)}$$

$L\left(\cosh t \int_0^t e^t \cosh t\, dt\right)$

$$= \frac{1}{2}\left[\frac{(s-2)}{(s-1)\left((s-2)^2 - 1\right)} + \frac{s}{(s+1)(s^2-1)}\right]$$

$$= \frac{1}{2}\left[\frac{(s-2)}{(s-1)(s^2 - 4s + 3)} + \frac{s}{(s+1)(s^2-1)}\right]$$