$$L\left(t\int_0^t e^{-2t}\sin 4t\,dt\right) = \frac{-d}{ds}\,\frac{4}{s^3+4s^2+20s}$$

$$= \frac{4(3s^2+8s+20)}{(s^3+4s^2+20s)^2}$$

Verify the following statement

$L(f'(t)) = sF(s) - f(0)$ where

$L(f(t)) = F(s)$ for $f(t) = 1-2e^{-3t}$

$$f(t) = 1-2e^{-3t}$$

$$f'(t) = -2(-3)e^{-3t} = 6e^{-3t}$$

$$LHS = L(f'(t)) = L(6e^{-3t}) = \frac{6}{s+3}.$$

$$F(s) = L(f(t)) = L(1-2e^{-3t})$$

$$= \frac{1}{s} - \frac{2}{s+3}$$

and $f(0) = 1-2e^{-0} = 1-2 = -1$

$$\therefore RHS = sF(s) - f(0)$$

$$= s\left(\frac{1}{s} - \frac{2}{s+3}\right) + 1$$

$$= 1 - \frac{2s}{s+3} + 1 = 2 - \frac{2s}{s+3} = \frac{6}{s+3}$$

$$L(f(t)) = L(\cos^2 3t)$$
$$= L\left(\frac{1+\cos 6t}{2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

i.e $$F(s) = \frac{1}{2s} + \frac{s}{2(s^2+36)}$$

$$f(0) = \cos^2 0 = 1$$

$$L(f'(t)) = sF(s) - f(0)$$
$$= s\left(\frac{1}{2s} + \frac{s}{2(s^2+36)}\right) - 1$$
$$= \frac{1}{2} + \frac{s^2}{2(s^2+36)} - 1$$
$$= \frac{s^2}{2(s^2+36)} - \frac{1}{2}$$

Using $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$

find $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$.

Let $f(t) = \sin\sqrt{t} \Rightarrow f(0) = \sin 0 = 0$

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L(f'(t)) = sF(s) - f(0)$$

$$L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) = s \cdot \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} - 0$$

$$\cancel{\tfrac{1}{2}} L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \cancel{\tfrac{1}{2}} \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Using Laplace transform, evaluate the following integrals :

1 $\int_0^\infty t e^{-3t} \sin t \, dt$

We will find $L(t e^{-3t} \sin t)$

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L(t \sin t) = -\frac{d}{ds} \frac{1}{(s^2+1)}$$

$$= + \frac{1 \times 2s}{(s^2+1)^2}$$

$$L(e^{-3t} t \sin t) = \frac{2(s+3)}{\{(s+3)^2 + 1\}^2}$$

$$\int_0^\infty e^{-st} \cdot e^{-3t} t \sin t \, dt = \frac{2(s+3)}{\{(s+3)^2+1\}^2}$$

Comparing with the integral that we want to evaluate,

put $s=0$

$$\int_0^\infty e^{-3t}\, t \sin t \, dt = \frac{2(3)}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$

2). $\displaystyle\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} \, dt$

To find $L\left(\frac{e^{-3t} - e^{-6t}}{t}\right)$

$$L(e^{-3t} - e^{-6t}) = \frac{1}{s+3} - \frac{1}{s+6}$$

$$L\left(\frac{e^{-3t} - e^{-6t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+6}\right) ds$$

$$= \Big[\log(s+3) - \log(s+6)\Big]_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+6}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1+3/s}{1+6/s}\right)\right]_s^\infty$$

$$= \log 1 - \log \frac{1+\frac{3}{s}}{1+\frac{6}{s}}$$

$$= 0 - \log \frac{s+3}{s+6} = \log \frac{s+6}{s+3}$$

$$L\left(\frac{e^{-3t} - e^{-6t}}{t}\right) = \log\left(\frac{s+6}{s+3}\right)$$

i.e $$\int_0^\infty e^{-st} \cdot \frac{e^{-3t} - e^{-6t}}{t}\, dt = \log\left(\frac{s+6}{s+3}\right)$$

Put $s = 0$

$$\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt = \log \frac{6}{3} = \log 2$$

3) $$\int_0^\infty \frac{\sin t}{t}\, dt$$

$$L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t}{t}\, dt = \frac{\pi}{2} - \tan^{-1} s$$

Put $s = 0$

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2} - \tan^{-1} 0 = \frac{\pi}{2}.$$