Theorem: The real and imaginary parts u and v of an analytic function $f(z) = u + iv$ are harmonic functions.

Since, $f(z)$ is an analytic function in some region, u and v satisfy the C-R equations

ie $u_x = v_y$ and $u_y = -v_x$

Differentiating $u_x = v_y$ wrt x

$$u_{xx} = v_{xy} \quad — ①$$

Differentiating $u_y = -v_x$ wrt y

$$u_{yy} = -v_{yx} \quad — ②$$

Assuming $v_{xy} = v_{yx}$ and adding ① and ②, we get

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0$$

Similarly we can show that

$v_{xx} + v_{yy} = 0$. This completes the proof.

Conversely, given a harmonic function $u(x,y)$ then prove that $f(z) = u_x - iu_y$ is an analytic function.

(This means given a harmonic f^n, it is the real part of an analytic function $f(z)$)

Given u is harmonic

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ — ①

Given $f(z) = u_x - i u_y$

$= U + iV$ say,

i.e $U = u_x$ and $V = -u_y$

$U_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$ — ②

$U_y = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}$ — ③

$V_x = \frac{\partial}{\partial x}\left(\frac{-\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial x \partial y}$ — ④

$V_y = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$ — ⑤

By ①, ② and ⑤ we get $U_x = V_y$

By ③, ④ we get $U_y = -V_x$

$\therefore f(z) = U + iV$ is analytic.

$\therefore f(z) = u_x - i u_y$ is analytic

Contrapositive:

I. Show that there does not exist an analytic function whose real part is

① $3x^2 + \sin x + y^2 + 5y + 4$

$u = 3x^2 + \sin x + y^2 + 5y + 4$

$u_x = 6x + \cos x$

$u_{xx} = 6 - \sin x$

$u_y = 2y + 5$

$u_{yy} = 2$

$u_{xx} + u_{yy} = 6 - \sin x + 2 = 8 - \sin x \neq 0$

Hence, not harmonic.
For an analytic f^n, both real and img parts are always harmonic.

② $u = x^2 + y^2$

$u_x = 2x$ $\quad$ $u_y = 2y$

$u_{xx} = 2$ $\quad$ $u_{yy} = 2$

$u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$

Hence not harmonic

For an analytic f^n, both real and imaginary parts are always harmonic.
As $u = x^2 + y^2$ is not harmonic, this can't be the real part of an analytic f^n.

Close connection between the Laplace eqn and analytic fn

I. Show that the following functions are harmonic and find their corresponding analytic fn $f(z) = u+iv$

① $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2=0$$

$\Rightarrow$ u is harmonic f^n

For $f(z) = u+iv$ to be analytic,

Using $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

$$f'(z) = u_x + iv_x$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x - i(-2y)$$

Using Milne-Thomson method, at $(z,0)$

$$f'(z) = 2z + i(0)$$

$$= 2z$$

$$f(z) = \frac{\cancel{2}z^2}{\cancel{2}} + c = z^2 + c$$

$$\boxed{f(z) = z^2 + c}$$

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(2) $u = 2x - 2xy$ Ans : $2z - iz^2 + c$

(3) $u = 3x^2y - y^3$ Ans : $z^3 + c$

(4) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ Ans : $z^3 + 3z^2 + c$

(5) $v = e^{2x}(y\cos 2y + x\sin 2y)$ Ans : $ze^{2z} + c$

III Show that the following f^n are harmonic and find their harmonic conjugate

(1) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\frac{\partial u}{\partial x} = 6xy + 4x$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

u satisfies Laplace's equation and hence harmonic.

Let $f(z) = u + iv$ be analytic f^n

$\therefore f'(z) = u_x + i v_x$

Using CR Eqn $u_x = v_y$ and $u_y = -v_x$

$\therefore f'(z) = u_x - i u_y$

$= (6xy + 4x) - i(3x^2 - 3y^2 - 4y)$

Using Milne Thomson method
Put $x = z$ and $y = 0$

$f'(z) = (6(0) + 4z) - i(3z^2 - 0)$

$= 4z - i3z^2$

Integrating,

$f(z) = 2z^2 - i z^3 + c$

Finding the harmonic conjugate

Put $z = x + iy$

$f(z) = 2(x+iy)^2 - i(x+iy)^3 + c$

$= 2(x^2 - y^2 + 2xyi)$

$- i(x^3 + i^3y^3 + 3x^2iy + 3x(iy)^2) + c$

$= 2(x^2 - y^2 + 2xyi) - i(x^3 - 3xy^2 + (3x^2y - y^3)i) + c$

$= 2(x^2 - y^2) + 4xyi - i(x^3 - 3xy^2) + (3x^2y - y^3) + c$

$= (2x^2 - 2y^2 + 3x^2y - y^3) + (4xy - x^3 + 3xy^2)i + c$

$\therefore$ Harmonic conjugate $v = 4xy - x^3 + 3xy^2 + c$

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(2) $u = e^x \cos y - x^2 + y^2$ Ans: $e^z - z^2 + c$

(3) $u = e^{2x}(y\cos 2y + x\sin 2y)$ Ans: $f(z) = z e^{2z} + c$

(4) $u = e^{-x}(x\cos y + y\sin y)$ Ans: $f(z) = i z e^{-z} + c$

Prove that $u = x^2 - y^2$ and $v = e^x \cos y$ both satisfy Laplace's equation but $u + iv$ is not an analytic fn.

Converse of [Theorem: If $f(z) = u + iv$ is analytic then its real and imaginary parts satisfy Laplace equation.] is not true.

For Any two harmonic functions u and v, $f(z) = u + iv$ may not be analytic.

Laplace's Equation in Polar Form:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

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Find the orthogonal trajectories of the family of the curves

① $3x^2y - y^3 = c$ Ans: $3xy^2 - x^3 = c$

② $x^2 - y^2 + x = c$ Ans $2xy + y = c$

③ $e^{-x}(x\sin y - y\cos y) = c$ Ans $e^{-x}(x\cos y + y\sin y) = c$

④ $x^2 - y^2 - 2xy + 2x - 3y = c$

Ans $2xy + 2y + x^2 - y^2 + 3y = c$

⑤ $e^x\cos y - xy = c$ Ans $e^x\sin y + \frac{x^2 - y^2}{2} = c$